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On the energetics of stratified turbulent mixing, irreversible thermodynamics, Boussinesq models and the ocean heat engine controversy

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In this paper, the available potential energy (APE) framework of Winters *et al.* (*J. Fluid Mech.*, vol. 289, 1995, p. 115) is extended to the fully compressible Navier–Stokes equations, with the aims of clarifying (i) the nature of the energy conversions taking place in turbulent thermally stratified fluids; and (ii) the role of surface buoyancy fluxes in the Munk & Wunsch (*Deep-Sea Res.*, vol. 45, 1998, p. 1977) constraint on the mechanical energy sources of stirring required to maintain diapycnal mixing in the oceans. The new framework reveals that the observed turbulent rate of increase in the background gravitational potential energy GPE_r, commonly thought to occur at the expense of the diffusively dissipated APE, actually occurs at the expense of internal energy, as in the laminar case. The APE dissipated by molecular diffusion, on the other hand, is found to be converted into internal energy (IE), similar to the viscously dissipated kinetic energy KE. Turbulent stirring, therefore, does not introduce a new APE/GPE_r mechanical-to-mechanical energy conversion, but simply enhances the existing IE/GPE_r conversion rate, in addition to enhancing the viscous dissipation and the entropy production rates. This, in turn, implies that molecular diffusion contributes to the dissipation of the available mechanical energy $ME = APE + KE$, along with viscous dissipation. This result has important implications for the interpretation of the concepts of mixing efficiency γ_{mixing} and flux Richardson number R_f, for which new physically based definitions are proposed and contrasted with previous definitions.

The new framework allows for a more rigorous and general re-derivation from the first principles of Munk & Wunsch (1998, hereafter MW98)'s constraint, also valid for a non-Boussinesq ocean:

$$G(KE) \approx \frac{1-\xi\, R_f}{\xi\, R_f} W_{r,forcing} = \frac{1+(1-\xi)\gamma_{mixing}}{\xi\, \gamma_{mixing}} W_{r,forcing},$$

where $G(KE)$ is the work rate done by the mechanical forcing, $W_{r,forcing}$ is the rate of loss of GPE_r due to high-latitude cooling and ξ is a nonlinearity parameter such that $\xi = 1$ for a linear equation of state (as considered by MW98), but $\xi < 1$ otherwise. The most important result is that $G(APE)$, the work rate done by the surface buoyancy fluxes, must be numerically as large as $W_{r,forcing}$ and, therefore, as important as the mechanical forcing in stirring and driving the oceans. As a consequence, the overall mixing efficiency of the oceans is likely to be larger than the value $\gamma_{mixing} = 0.2$ presently used, thereby possibly eliminating the apparent shortfall in mechanical stirring energy that results from using $\gamma_{mixing} = 0.2$ in the above formula.

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1. Introduction

1.1. *Stirring versus mixing in turbulent stratified fluids*

As is well known, stirring by the velocity field greatly enhances the amount of irreversible mixing due to molecular diffusion in turbulent stratified fluid flows, as compared with the laminar case. A rigorous proof of this result exists for thermally driven Boussinesq fluids for which boundary conditions are either of no-flux or fixed temperature. In that case, it is possible to show that

$$\Phi = \frac{\displaystyle\int_V \|\nabla T\|^2 \mathrm{d}V}{\displaystyle\int_V \|\nabla T_c\|^2 \mathrm{d}V}, \tag{1.1}$$

i.e. the ratio of the entropy production (in the Boussinesq limit) of the stirred state over that of the corresponding purely conductive non-stirred state is always greater than unity, where T and T_c are the temperature of the stirred and conductive states respectively, the proof being originally due to Zeldovich (1937) and re-derived by Balmforth & Young (2003). The function Φ was introduced by Paparella & Young (2002) as a measure of the strength of the circulation driven by surface buoyancy fluxes. However, because Φ is analogous to an average Cox number (the local turbulent effective diffusivity normalized by the background diffusivity; e.g. Osborn & Cox 1972; Gregg 1987), it is also representative of the amount of turbulent diapycnal mixing taking place in the fluid.

Reversible stirring and irreversible mixing (see, e.g. Eckart 1948) occur in relation to physically distinct types of forces at work in the fluid. Stirring works against buoyancy forces by lifting and pulling relatively heavier and lighter parcels respectively, thus causing a reversible conversion between kinetic energy (KE) and available potential energy (APE). Mixing, on the other hand, is the byproduct of the work done by the generalized thermodynamic forces associated with molecular viscous and diffusive processes that relax the system towards thermodynamic equilibrium (see, e.g. de Groot & Mazur 1962; Kondepudi & Prigogine 1998; Ottinger 2005). Thus, stirring enhances the work rate done by the viscous stress against the velocity field, resulting in enhanced dissipation of KE into internal energy (IE). Similarly, stirring also enhances the thermal entropy production rate associated with the heat transfer imposed by the second law of thermodynamics, which results in a diathermal effective diffusive heat flux that is increased by the ratio $(A_{turbulent}/A_{laminar})^2$ (another measure of the Cox number), where $A_{turbulent}$ and $A_{laminar}$ refer to the 'turbulent' and 'laminar' areas of a given isothermal surface (see Nakamura 1996; Winters & d'Asaro 1996). In the laminar regime, the generalized thermodynamic forces associated with molecular diffusion are known to cause the conversion of IE into background gravitational potential energy (GPE_r). From a thermodynamic viewpoint, it would be natural to expect the stirring to enhance the IE/GPE_r conversion, but in fact, the existing literature usually accounts for the observed turbulent increase in GPE_r as the result of a 'new' energy conversion irreversibly converting APE into GPE_r. Clarifying this controversial issue is a key objective of this paper.

1.2. *The modern approach to the energetics of turbulent mixing*

The most rigorous existing theoretical framework for understanding the interactions between the different forces at work in a turbulent stratified fluid is probably the available potential energy framework introduced by Winters *et al.* (1995); it is so far the only framework that rigorously separates reversible effects due to stirring from the irreversible effects due to mixing (see also Tseng & Ferziger 2001). As originally proposed by Lorenz (1955), such a framework separates the potential energy PE (i.e. the sum of the GPE andthe IE) into its available ($APE = AGPE + AIE$) and non-available ($PE_r = GPE_r + IE_r$) components, with the IE component being neglected for a Boussinesq fluid, the case considered by Winters *et al.* (1995). The usefulness of such a decomposition stems from the fact that the background reference state is by construction affected only by diabatic and/or irreversible processes, so that understanding how the reference state evolves provides insight into how much mixing takes place in the fluid.

In the case of a freely decaying turbulent Boussinesq stratified fluid with an equation of state linear in temperature, referred to as the L-Boussinesq model hereafter, Winters *et al.* (1995) show that the evolution equations for KE, $APE = AGPE$ and GPE_r take the form:

$$\frac{\mathrm{d}\,KE}{\mathrm{d}t} = -C(KE, APE) - D(KE), \tag{1.2}$$

$$\frac{\mathrm{d}\,APE}{\mathrm{d}t} = C(KE, APE) - D(APE), \tag{1.3}$$

$$\frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} = W_{r,mixing} = W_{r,turbulent} + W_{r,laminar}, \tag{1.4}$$

where $C(APE, KE) = -C(KE, APE)$ is the so-called buoyancy flux measuring the reversible conversion between KE and APE, $D(APE)$ is the diffusive dissipation of APE, which is related to the dissipation of temperature variance χ (e.g. Holloway 1986; Zilitinkevich *et al.* 2008), while $W_{r,mixing}$ is the rate of change in GPE_r induced by molecular diffusion, which is commonly decomposed into a laminar $W_{r,laminar}$ and a turbulent $W_{r,turbulent}$ contribution. All these terms are explicitly defined in Appendix A for the L-Boussinesq model, as well as for a Boussinesq fluid whose thermal expansion increases with temperature, called the NL-Boussinesq model. Appendix B further generalizes the corresponding expressions for the fully compressible Navier–Stokes equations (CNSE) with an arbitrary nonlinear equation of state (depending on pressure and temperature only, though).

Of particular interest in turbulent mixing studies is the behaviour of $W_{r,turbulent}$ – the turbulent rate of increase in GPE_r – which has been mostly discussed in the context of the L-Boussinesq model, for which an important result is

$$W_{r,turbulent} = D(APE), \tag{1.5}$$

which states the equality between the APE dissipation rate and $W_{r,turbulent}$. This result is important, because from the known properties of $D(APE)$, it is clear that enhanced diapycnal mixing rates fundamentally require: (i) finite values of APE, since $D(APE) = 0$ when $APE = 0$; and (ii) an APE cascade transferring the spectral energy of the temperature (density) field to the small scales at which molecular diffusion is the most efficient at smoothing out temperature gradients. The discussion of the APE cascade, which is closely related to that of the temperature variance, has an extensive literature related to explaining the k^{-3} spectra in the so-called buoyancy subrange, both in the atmosphere (e.g. Lindborg 2006) and in the oceans (e.g. Holloway 1986; Bouruet-Aubertot, Sommeria & Staquet 1996). Note that because APE is a globally defined scalar quantity, speaking of APE cascades requires the introduction of the so-called APE density, noted $\Phi_a(\boldsymbol{x}, t)$ here, for which a spectral description is possible

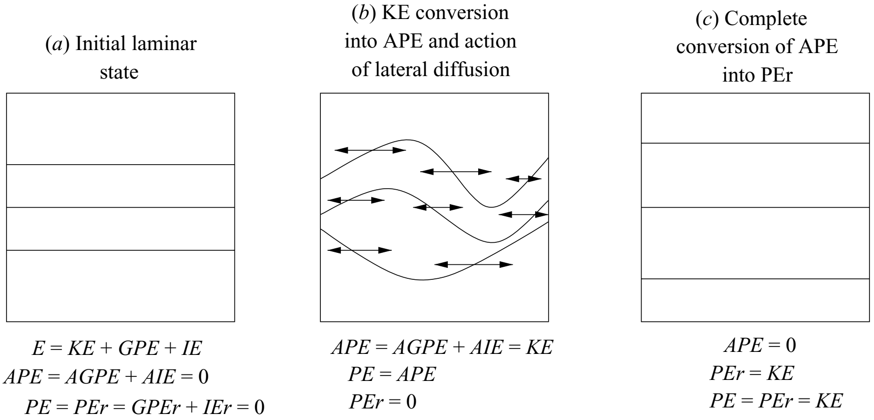


FIGURE 1. Idealized depiction of the diffusive route for kinetic energy dissipation. (*a*) The laminar state possessing initially no AGPE and AIE, but some amount of KE. (*b*) The state obtained by the reversible adiabatic conversion of some kinetic energy into APE, which increases APE but leaves the background GPE_r and IE_r unchanged. (*c*) The state obtained by letting the horizontal part of molecular diffusion smooth out the isothermal surfaces until all the APE in (*b*) have been converted into background $PE_r = GPE_r + IE_r$.

(see, e.g. Holliday & McIntyre 1981; Roullet & Klein 2009; Molemaker & McWilliams 2009).

Equations (1.2)–(1.4) exhibit only one type of reversible conversion, namely the 'buoyancy flux' associated with the APE/KE conversion, and three irreversible conversions – $D(KE)$, $D(APE)$ and $W_{r,mixing}$ – the first one caused by molecular viscous processes and the latter two caused by molecular diffusive processes. The primary goal of turbulence theory is to understand how the reversible $C(APE, KE)$ conversion and irreversible $D(KE)$, $D(APE)$ and $W_{r,mixing}$ are all interrelated. In this paper, we focus on turbulent diffusive mixing, for the understanding of viscous dissipation constitutes somehow a separate issue with its own problems (e.g. Gregg 1987). The nature of these links is usually explored by estimating the energy budget of a turbulent mixing event, defined here as a period of intense mixing preceded and followed by laminar conditions, for which there is a huge literature of observational, theoretical and numerical studies. Integrating the above energy equations over the duration of the turbulent mixing event yields

$$\Delta KE = -\overline{C(KE, APE)} - \overline{D(KE)}, \tag{1.6}$$

$$\Delta APE = \overline{C(KE, APE)} - \overline{D(APE)}, \tag{1.7}$$

$$\Delta GPE_r = \overline{W}_{r,mixing} = \overline{W}_{r,turbulent} + \overline{W}_{r,laminar}, \tag{1.8}$$

where $\Delta(.)$ and the overbar denote respectively the net variation and the time-integral of a quantity over the mixing event. Summing the KE and APE equations yields the important 'available' mechanical energy equation

$$\Delta KE + \Delta APE = -[\overline{D(KE)} + \overline{D(APE)}] < 0, \tag{1.9}$$

which states that the total 'available' mechanical energy, $ME = KE + APE$, undergoes a net decrease over the mixing event as the result of the viscous and diffusive dissipation of KE and APE, respectively. A schematic of the APE dissipation process, which provides a diffusive route to KE dissipation, is illustrated in figure 1.

1.3. *Measures of mixing efficiency in turbulent stratified fluids*

Equation (1.9) makes it clear that turbulent diapycnal mixing (through $D(APE)$) participates in the total dissipation of available mechanical energy $ME = KE + APE$. Since $D(APE)$ is non-zero only if APE is non-zero, turbulent diapycnal mixing requires having as much of ME in the form of APE as possible. The classical concept of 'mixing efficiency', reviewed below, seeks to provide a number quantifying the ability of a particular turbulent mixing event in dissipating $ME = KE + APE$ preferentially diffusively rather than viscously. From a theoretical viewpoint, it is useful to separate turbulent mixing events into two main archetypal categories, corresponding to the two cases where ME is initially entirely in either KE or APE form. These two cases are treated separately before providing a synthesis addressing the general case.

At a fundamental level, quantifying the mixing efficiency of a turbulent mixing event requires two numbers: one to measure how much of ME is viscously dissipated and the other to measure how much of ME is dissipated by turbulent mixing. While everybody seems to agree that $\overline{D(KE)}$ is the natural measure of viscous dissipation, it is the buoyancy flux $\overline{C(APE, KE)}$, rather than $D(APE)$, that has been historically thought to be the relevant measure of how much of ME is dissipated by turbulent mixing, since it is the term in (1.6) that seems to be removing KE along with viscous dissipation. For mechanically driven turbulent mixing events, defined here such that $\Delta APE = 0$ and $\Delta ME = \Delta KE$, the efficiency of mixing has been classically quantified by two important numbers. The first is the so-called flux Richardson number R_f, defined by Linden (1979) as 'the fraction of the change in available kinetic energy which appears as the potential energy of the stratification', mathematically defined as

$$R_f = \frac{\overline{C(KE, APE)}}{|\Delta KE|} = \frac{\overline{C(KE, APE)}}{\overline{C(KE, APE)} + \overline{D(KE)}} \tag{1.10}$$

(see Osborn 1980), and the second is the so-called mixing efficiency:

$$\gamma_{mixing} = \frac{R_f}{1 - R_f} = \frac{\overline{C(KE, APE)}}{\overline{D(KE)}}. \tag{1.11}$$

It is now recognized, however, that the buoyancy flux represents only an indirect measure of irreversible mixing, since it physically represents a reversible conversion between KE and APE, while furthermore appearing to be difficult to interpret empirically (see, e.g. Barry *et al.* 2001 and references therein). Recognizing this difficulty, Caulfield & Peltier (2000) and Staquet (2000) effectively suggested to replace $C(\overline{KE, APE})$ by a more direct measure of irreversible mixing in the above definitions of R_f and γ_{mixing}. Since turbulent diapycnal mixing is often diagnosed empirically from measuring the net changes in GPE_r over a mixing event (e.g. (McEwan 1983*a*, 1983*b*; Barry *et al.* 2001; Dalziel *et al.* 2008), a natural choice is to use $\overline{W}_{r,turbulent}$ as a direct measure of irreversible mixing, which leads to

$$R_f^{GPE_r} = \frac{\overline{W}_{r,turbulent}}{\overline{W}_{r,turbulent} + \overline{D(KE)}}, \tag{1.12}$$

$$\gamma_{mixing}^{GPE_r} = \frac{R_f^{GPE_r}}{1 - R_f^{GPE_r}} = \frac{\overline{W}_{r,turbulent}}{\overline{D(KE)}}. \tag{1.13}$$

From a theoretical viewpoint, these definitions are justified from the fact that in the L-Boussinesq model, the following equalities hold:

$$\overline{C(APE, KE)} = \overline{D(APE)} = \overline{W}_{r,turbulent}, \tag{1.14}$$

as follows from (1.6) and (1.7), combined with (1.5), when $\Delta APE = 0$. The modified flux Richardson number $R_f^{GPE_r}$ coincides – for a suitably defined time interval – with the cumulative mixing efficiency $\mathscr{E}_c$ introduced by Caulfield and Peltier (2000), as well as with the generalized flux Richardson number R_b defined by Staquet (2000), in which our $\gamma_{mixing}^{GPE_r}$ is also denoted by γ_b.

Although (1.12) and (1.13) are consistent with the traditional buoyancy–flux-based definitions of R_f and γ_{mixing} in the context of the L-Boussinesq model, such definitions overlook the fact that (1.14) is not valid in the more general context of the fully compressible Navier–Stokes equations, for which the ratio

$$\xi = \frac{W_{r,turbulent}}{D(APE)} \tag{1.15}$$

is in general less than one, and even sometimes negative, for water or seawater. For this reason, it appears that R_f and γ_{mixing} should, in fact, be defined in terms of $D(APE)$, not $W_{r,turbulent}$, viz.,

$$R_f^{DAPE} = \frac{\overline{D(APE)}}{\overline{D(APE)} + \overline{D(KE)}}, \tag{1.16}$$

$$\gamma_{mixing}^{DAPE} = \frac{\overline{D(APE)}}{\overline{D(KE)}}, \tag{1.17}$$

which we call the dissipation flux Richardson number and the dissipation mixing efficiency respectively, to distinguish them from their predecessors. In our opinion, R_f^{DAPE} and γ_{mixing}^{DAPE} as defined by (1.16) and (1.17) are really the ones that are truly consistent with the properties assumed to be attached to those numbers. Most notably, (1.16) is the only way to define a flux Richardson number that is guaranteed to lie within the interval [0, 1], since neither $\overline{C(KE, APE)}$ nor $\overline{W}_{r,turbulent}$ can be ascertained to be positive under all circumstances. Since (1.12) and (1.13) are still likely to be used in the future owing to their practical interest, it is useful to provide conversion rules between the GPE_r and $D(APE)$-based definitions of R_f and γ_{mixing}, viz.,

$$\gamma_{mixing}^{GPE_r} = \xi \gamma_{mixing}^{DAPE}, \qquad R_f^{GPE_r} = \frac{\xi R_f^{DAPE}}{1-(1-\xi)R_f^{DAPE}}. \tag{1.18}$$

These formulae require knowledge of the nonlinearity parameter ξ, which measures the importance of nonlinear effects associated with the equation of state (see Tailleux 2009 for details). The often-cited canonical value for mechanically driven turbulent mixing is $\gamma_{mixing} \approx 0.2$, which appears to date back from Osborn (1980) (e.g. Peltier & Caulfield 2003).

The second case of interest, namely buoyancy-driven turbulent mixing, is defined here as being such that $\Delta KE = 0$ and $\Delta ME = \Delta APE$, as occurs in relation to the so-called Rayleigh–Taylor instability for instance. Equations (1.6) and (1.7) lead to

$$\overline{C(KE, APE)} = -\overline{D(KE)} < 0, \tag{1.19}$$

$$\overline{D(APE)} = \overline{C(KE, APE)} - \Delta APE = |\Delta APE| - |\overline{C(KE, APE)}|. \tag{1.20}$$

Equation (1.19) reveals that the buoyancy flux is negative this time and it represents the fraction of ME that is lost to *viscous* dissipation, *not* diffusive dissipation. This establishes, if needed, that the buoyancy flux should not be systematically interpreted as a measure of irreversible diffusive mixing. Since Linden (1979)'s above definition for the flux Richardson number does not really make sense for Rayleigh–Taylor instability, an alternative definition is called for. The most natural definition, in our opinion, is as the fraction of ME dissipated by irreversible diffusive mixing, viz.,

$$R_f = \frac{-\Delta APE + \overline{C(KE, APE)}}{-\Delta APE} = 1 - \frac{\overline{|C(KE, APE)|}}{|\Delta APE|}, \tag{1.21}$$

which, according to (1.6) and (1.7), is equivalent to

$$R_f = \frac{\overline{D(APE)}}{\overline{D(APE)} + \overline{D(KE)}}, \tag{1.22}$$

with the corresponding value of γ_{mixing}:

$$\gamma_{mixing} = \frac{R_f}{1 - R_f} = \frac{\overline{D(APE)}}{\overline{D(KE)}}, \tag{1.23}$$

which are identical to (1.16) and (1.17). The above results make it possible, therefore, to use R_f^{DAPE} and γ_{mixing}^{DAPE} as definitions for the flux Richardson number and mixing efficiency that make sense for all possible types of turbulent mixing events.

At this point, a note about terminology seems to be warranted, since in the case of the Rayleigh–Taylor instability, it is R_f that is referred to as the mixing efficiency by some authors (e.g. Linden & Redondo 1991; Dalziel *et al.* 2008), rather than γ_{mixing}. Physically, this seems more logical, since R_f is always comprised within the interval [0, 1], whereas γ_{mixing} is not. Interestingly, Oakey (1982) appears to be the first to define γ_{mixing} as a 'mixing coefficient representing the ratio of potential energy to kinetic energy dissipation'. For this reason, it would seem more appropriate and less ambiguous to refer to γ_{mixing} as the 'dissipations ratio'. Unfortunately, it is not always clear in the literature which quantity the widely used term 'mixing efficiency' refers to, as it has been used so far to refer to both R_f and γ_{mixing}. In order to avoid ambiguities, the remaining paper only makes use of the quantities R_f^{DAPE} and γ_{mixing}^{DAPE}, which for simplicity are denoted by R_f and γ_{mixing}, respectively.

As a side note, it seems important to point out that Rayleigh–Taylor instability has the peculiar property that $\Delta GPE_{r,max}$, the maximum possible increase in GPE_r achieved for the fully homogenized state, is only *half* the initial amount of APE (at least when $\xi = 1$, i.e. in the context of the L-Boussinesq model; e.g. Linden & Redondo 1991; Dalziel *et al.* 2008). Physically, it means that less than 50 % of the initial APE can actually contribute to turbulent diapycnal mixing, and hence that at least 50 % of it must be eventually viscously dissipated. As a result, one has the following constraints:

$$R_f = \frac{\overline{D(APE)}}{|\Delta APE|} = \frac{\xi \overline{W}_{r,turbulent}}{|\Delta APE|} \leqslant \frac{1}{2}, \tag{1.24}$$

$$\gamma_{mixing} \leqslant \frac{\xi/2}{1 - \xi/2} \leqslant 1. \tag{1.25}$$

Experimentally, Linden & Redondo (1991) reported values of $R_f \approx 0.3$ ($\gamma_{mixing} = 3/7 \approx 0.43$), while Dalziel *et al.* (2008) reported experiments in which the maximum possible value $R_f \approx 0.5$ ($\gamma_{mixing} \approx 1$) was reached. Owing to the peculiarity of the Rayleigh–Taylor instability, however, one should refrain from concluding that $\gamma_{mixing} = 1$ or $R_f = 0.5$ represents the maximum possible values for γ_{mixing} and R_f in turbulent stratified fluids. To reach definite and general conclusions about γ_{mixing} and R_f, more general examples of buoyancy-driven turbulent mixing events should be studied. It would be interesting, for instance, to study the mixing efficiency of

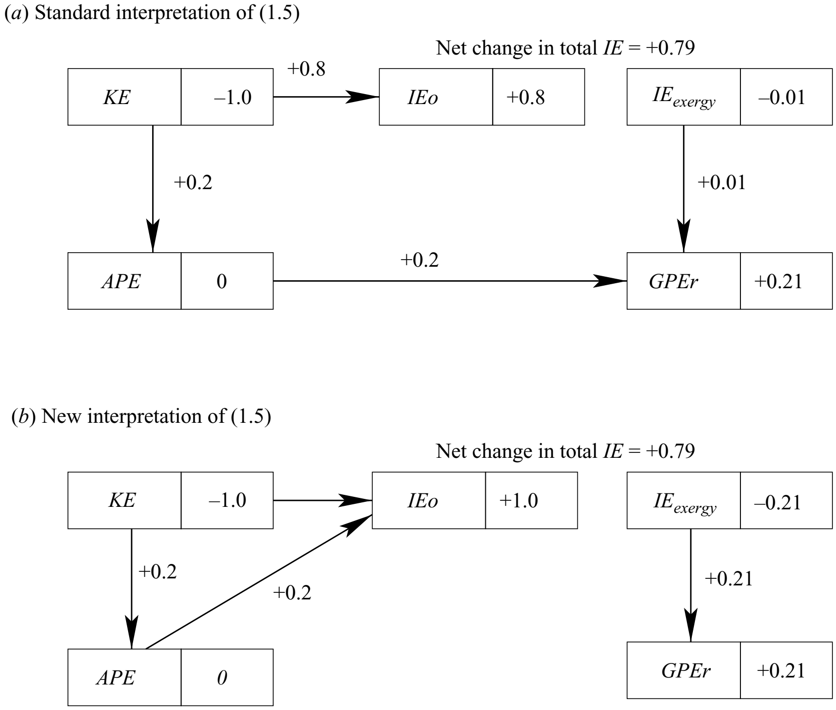


FIGURE 2. (*a*) Predicted energy changes for a hypothetical turbulent mixing event under the assumption that the diffusively dissipated APE is irreversibly converted into GPE_r. (*b*) Same as in (*a*) under the assumption that the diffusively dissipated APE is irreversibly converted into IE_0, as the viscously dissipated KE. In both cases, the net energy changes in KE, GPE_r, APE and IE are the same. The only predicted differences concern the subcomponents of the internal energy IE_0 and IE_{exergy}.

a modified Rayleigh–Taylor instability such that the unstable stratification occupies only half or less of the spatial domain, so that $\Delta GPE_{r,max} \geqslant |\Delta APE|$. In this case, all of the initial APE could, in principle, be dissipated by molecular diffusion, which would correspond to the limits $R_f = 1$ and $\gamma_{mixing} = +\infty$. Of course, such limits cannot be reached, as it is impossible to prevent part of the APE to be converted into KE, part of which will necessarily be dissipated viscously, but they are nevertheless important in suggesting that values of $\gamma_{mixing} > 1$ can, in principle, be reached, which sets an interesting goal for future research.

1.4. *On the nature of* $D(APE)$ *and* $W_{r,turbulent}$

Of fundamental importance in understanding the physics of turbulent diapycnal mixing are the nature and type of the energy conversions associated with $D(APE)$ and $W_{r,turbulent}$. So far, it seems fair to say that these two energy conversions have been regarded as essentially being one and the same, based on the exact equality $W_{r,turbulent} = D(APE)$ occurring in the L-Boussinesq model, suggesting that molecular diffusion irreversibly converts APE into GPE_r (e.g. Winters *et al.* 1995). Such an interpretation now appears to be widely accepted (e.g. MW98; Caulfield & Peltier 2000; Peltier & Caulfield 2003; Huang 2004; Thorpe 2005, among others). The main characteristic of this view, schematically illustrated in figure 2(*a*), is to disregard the possibility that the turbulent increase of GPE_r might be due to the enhancement of the IE/GPE_r conversion rate by the stirring. In other words, the current view assumes that the work involved in the turbulent increase of GPE_r is done by the stirring against buoyancy forces, not by the generalized thermodynamic forces responsible for entropy production and the IE/GPE_r conversion. At the same time, the current view seems to accept that stirring enhances entropy production. But from classical thermodynamics, this seems possible only if the work rate done by the generalized thermodynamic forces is also enhanced, which in turn should imply an enhanced IE/GPE_r conversion.

In order to determine whether the turbulent increase of GPE_r could be accounted for by a stirring-enhanced IE/GPE_r conversion rate, rather than by the irreversible conversion of APE into GPE_r, it is useful to point out that the validity of Winters *et al.* (1995)'s interpretation seems to rely crucially on $D(APE)$ and $W_{r,turbulent}$ being *exactly* identical, not only mathematically (as is the case in the L-Boussinesq model) but also physically. Here, two quantities are defined as being physically equal if they remain mathematically equal in more accurate models of fluid flows – closer to physical 'truth' in some sense – such as CNSE. Indeed, only a physical equality can define a physically valid energy conversion, as we hope the reader will agree. However, as shown in Appendix B, which extends Winters *et al.* (1995) results to the CNSE, the equality $D(APE)=W_{r,turbulent}$ is found to be a serendipitous feature of the L-Boussinesq model, which at best is only a good approximation, the general result being that the ratio

$$\xi = \frac{W_{r,turbulent}}{D(APE)} \tag{1.26}$$

usually lies within the interval $-\infty < \xi < 1$ for water or seawater and it strongly depends on the nonlinear character of the equation of state. Whether there exist fluids allowing for $\xi > 1$ is not known yet. An important result is that it appears to be perfectly possible for GPE_r to *decrease* as the result of turbulent mixing, in contrast to what is often stated in the literature. This case, which corresponds to $\xi < 0$, was in fact previously identified and discussed by the late Nick Fofonoff in a series of little known papers (see Fofonoff 1962, 1998, 2001). For this reason, the case $\xi < 0$ shall be subsequently referred to as the *Fofonoff regime*, while the more commonly studied case for which $W_{r,turbulent} > 0$ shall be referred to as the *classical regime*.

The lack of physical equality between $D(APE)$ and $W_{r,turbulent}$ makes Winters *et al.* (1995)'s interpretation very unlikely and gives strong credence to the idea that $W_{r,turbulent}$ actually corresponds to a stirring-enhanced IE/GPE_r conversion rate. If so, what about $D(APE)$? In order to shed light on the issue of APE dissipation, it is useful to recall some well-known properties of thermodynamic transformations associated with the following problem: Assuming that the potential energy $PE = GPE + IE$ of a stratified fluid increases by ΔE, how is ΔE split between ΔGPE and ΔIE? Here, standard thermodynamics tells us that the answer depends on whether ΔE is added reversibly or irreversibly to PE. Thus, if ΔE is added reversibly to PE (i.e. without entropy change, and for a nearly incompressible fluid), then

$$\frac{\Delta GPE}{\Delta E} \approx 1, \qquad \frac{\Delta IE}{\Delta E} \ll 1, \tag{1.27}$$

while if ΔE is added irreversibly (i.e. with an increase in entropy), then

$$\frac{\Delta GPE}{\Delta E} \ll 1, \qquad \frac{\Delta IE}{\Delta E} \approx 1, \tag{1.28}$$

i.e. the opposite. These results, therefore, suggest that when molecular diffusion converts APE into PE_r, the dissipated APE must nearly entirely go into IE_r, not

GPE_r, in contrast to what is usually assumed. (The demonstration of (1.27) and (1.28) is omitted for brevity, but this follows from the results of Appendix B.) It follows that what the equality $D(APE) = W_{r,turbulent}$ of the L-Boussinesq actually states is the equality of the APE/IE and IE/GPE_r conversion rates (or more generally, for real fluids, the correlation between the two rates), not that $D(APE)$ and $W_{r,turbulent}$ are of the same type. Physically, the two conversion rates $W_{r,turbulent}$ and $D(APE)$ appear to be fundamentally correlated because they are controlled by both molecular diffusion and the spectral distribution of APE, as will be made clear later in the text.

1.5. *Internal energy or internal energies?*

In the new interpretation proposed above, internal energy is destroyed by the IE/GPE_r conversion at the turbulent rate $W_{r,turbulent}$, while being created by the APE dissipation at the turbulent rate $D(APE)$. Could it be possible, therefore, for the dissipated APE to be eventually converted into GPE_r, not by the direct APE/GPE_r conversion route proposed by Winters *et al.* (1995), as this was ruled out by thermodynamic considerations, but indirectly by transiting through the IE reservoir?

As shown in Appendix B, the answer to the above question is found to be negative, because it turns out that the kind of IE which APE is dissipated into appears to be different from the kind of IE being converted into GPE_r. Specifically, Appendix B shows that IE is indeed best regarded as the sum of distinct sub-reservoirs. In this paper, three such sub-reservoirs are introduced: the available internal energy (AIE), the exergy (IE_{exergy}) and the dead internal energy (IE_0). Physically, this decomposition parallels the following temperature decomposition: $T(x,y,z,t) = T'(x,y,z,t) + T_r(z,t) + T_0(t)$, where $T_0(t)$ is the equivalent thermodynamic equilibrium temperature of the system, $T_r(z,t)$ is Lorenz's reference vertical temperature profile and $T'(x,y,z,t)$ is the residual. Physically, AIE is the internal energy component of Lorenz (1955)'s APE, while IE_0 and IE_{exergy} are the internal energy associated with the equivalent thermodynamic equilibrium temperature T_0 and vertical temperature stratification T_r, respectively. The idea behind this decomposition can be traced back to Gibbs (1878), the concept of exergy being common in the thermodynamic engineering literature (e.g. Bejan 1997). See also Marquet (1991) for an application of exergy in the context of atmospheric available energetics. A full review of existing ideas related to the present ones is beyond the scope of this paper, as the engineering literature about available energetics and exergy is considerable. The way it works is encompassed in the following equations:

$$\frac{\mathrm{d}KE}{\mathrm{d}t} = -C(KE, APE) - D(KE), \tag{1.29}$$

$$\frac{\mathrm{d}APE}{\mathrm{d}t} = C(KE, APE) - D(APE), \tag{1.30}$$

$$\frac{\mathrm{d}GPE_r}{\mathrm{d}t} = W_{r,mixing} = W_{r,laminar} + W_{r,turbulent}, \tag{1.31}$$

$$\frac{\mathrm{d}IE_0}{\mathrm{d}t} \approx D(KE) + D(APE) = D_{total}, \tag{1.32}$$

$$\frac{\mathrm{d}IE_{exergy}}{\mathrm{d}t} \approx -W_{r,mixing} = -W_{r,laminar} - W_{r,turbulent}. \tag{1.33}$$

In this model, the first three equations are just a rewriting of (1.2) and (1.4), so that the main novelty is associated with (1.32) and (1.33). Physically, (1.32) states that the viscous and diffusive dissipation processes $D(KE)$ and $D(APE)$ mostly affect T_0 but not T_r, while (1.33) states that the IE/GPE_r conversion reduces IE_{exergy} as well as smoothes out T_r. The empirical verification of the validity of the above equations is the main topic of § 2.

1.6. *Link with the ocean heat engine controversy*

In the oceans, turbulent diapycnal mixing is a crucial process, as it is required to transport heat from the surface equatorial regions down to the depths cooled by high-latitude deep-water formation. In the traditional picture found in most oceanography textbooks, turbulent diapycnal mixing and deep water formation are usually described as part of the buoyancy-driven component of the large-scale ocean circulation responsible for the oceanic poleward transport of heat, often called the meridional overturning circulation (MOC). Physically, the MOC is often equated with the longitudinally averaged circulation taking place in the latitudinal/vertical plane. The possible dependence of the buoyancy-driven circulation on mechanical forcing, which one might expect in a system as nonlinear as the oceans, has been usually ignored. However, the idea of a buoyancy-driven circulation unaffected by mechanical forcing physically makes sense only if one can establish that the mechanical stirring required to sustain turbulent diapycnal mixing is driven by surface buoyancy fluxes. MW98 questioned this view and argued instead that turbulent diapycnal mixing must, in fact, be primarily driven by the wind and tides, and hence that the buoyancy-driven circulation must in fact be mechanically controlled. Moreover, MW98 analysed the GPE budget of the oceans to derive the following constraint:

$$G(KE) = \frac{W_{r,forcing}}{\gamma_{mixing}}, \tag{1.34}$$

linking the work rate $G(KE)$ done by the mechanical sources of stirring, the rate $W_{r,forcing}$ at which high-latitude cooling depletes GPE_r, and the oceanic bulk 'mixing efficiency' γ_{mixing} (or more accurately, the dissipations ratio, as argued previously). Physically, (1.34) states that the fraction γ_{mixing} of $G(KE)$ has to be expanded into turbulent mixing to raise GPE_r at the same rate $W_{r,forcing}$ at which it is lost. By using the values $W_{r,forcing} \approx 0.4\,\mathrm{TW}$ and $\gamma_{mixing} = 0.2$, MW98 concluded that $G(KE) = O(2\,\mathrm{TW})$ is approximately required to sustain the observed oceanic rates of turbulent diapycnal mixing. This result caused much stirring in the ocean community, because the wind supplies only about 1 TW, leaving an apparent shortfall of 1 TW to close the energy budget. This led MW98 to argue that the only plausible candidate to account for the missing stirring should be the tides, spawning a considerable research effort over the past 10 years on the issue of tidal mixing.

Although MW98's arguments have been echoed favourably within the ocean community (see, e.g. Paparella & Young 2002; Huang 2004; Wunsch & Ferrari 2004; Kuhlbrodt *et al.* 2007; Nycander *et al.* 2007), it remains unclear why the surface buoyancy fluxes should not be important in stirring and driving the oceans, given that the work rate done by the surface buoyancy fluxes, as measured by the APE production rate, was previously estimated by Oort, Anderson & Peixoto (1994) to be $G(APE) = 1.2 \pm 0.7\,\mathrm{TW}$ and hence comparable in importance with the work rate done by the mechanical forcing. In their paper, MW98 rather summarily dismissed Oort *et al.* (1994)'s results by contending that the so-called Sandström (1908)'s theorem requires that $G(APE)$ be negligible, and hence that the buoyancy forcing cannot produce any significant work in the oceans, but given the highly controversial nature of Sandström (1908)'s paper in the physical oceanography community and its apparent refutation by Jeffreys (1925), it is important to have a more solid physical basis to make any definitive statements about $G(APE)$. Note that Sandstrom's paper was recently translated by Kuhlbrodt (2008), who argues that Sandstrom did not initially formulate his results as a theorem, but rather as an inference. Ascertaining whether $G(APE)$ is large or small is obviously crucial in determining whether the MOC is effectively driven by the turbulent mixing powered by the winds and tides, as argued by MW98, or whether it is in fact predominantly buoyancy-driven, as it appears to be possible if $G(APE)$ is as large as predicted by Oort *et al.* (1994). Further clarification is also needed to understand the possible importance of some effects neglected by MW98, such as those due to a nonlinear equation of state, which Gnanadesikan *et al.* (2005) argue lead to a significant underestimation of $G(KE)$, or due to entrainment effects, which Hughes & Griffiths (2006) argue lead to a possible significant overestimation of $G(KE)$.

1.7. *Purpose and organization of the paper*

The primary objective of this paper is to clarify the nature of the energy conversions taking place in turbulent stratified fluids, with the aim of clarifying the underlying assumptions entering MW98's energy constraint equation (1.34). The backbone of the paper are the theoretical derivations presented in Appendices A and B, which provide a rigorous theoretical support to understand the links between stirring and irreversible mixing in mechanically and thermodynamically forced thermally stratified fluids. Appendix A offers a new derivation of Winters *et al.* (1995)'s framework, which is further extended to the case of a Boussinesq fluid with a thermal expansion increasing with temperature. Appendix B is an extension to the case of a fully compressible thermally stratified fluid, in which the decomposition of internal energy into three distinct sub-reservoirs is presented. Section 2 illustrates the differences between $D(APE)$ and $W_{r,turbulent}$ using a number of different viewpoints and examines some of its consequences, in the context of freely decaying turbulence. Section 3 revisits the issues pertaining to MW98's energy constraint. Section 4 offers a summary and discussion of the results.

2. A new view of turbulent mixing energetics in freely decaying stratified turbulence

2.1. *Boussinesq versus non-Boussinesq energetics*

As mentioned above, a central point of this paper is to argue that irreversible energy conversions in turbulent stratified fluids are best understood if internal energy is not regarded as a single energy reservoir, but as the sum of at least three distinct sub-reservoirs. Obviously, these nuances are lost in the traditional Boussinesq description of turbulent fluids, since the latter lacks an explicit representation of internal energy, let alone of its three sub-reservoirs. This does not mean that the Boussinesq approximation is necessarily inaccurate or incomplete, but rather that the definitive interpretation of its energy conversions requires to be checked against the understanding gained from the study of the fully compressible Navier–Stokes equations. Such a study was carried out and its results are reported in Appendix B. In our approach, successive refinements of the energy conversions were sought, starting from the $KE/APE/PE_r$ system for which the number of energy conversions is limited and unambiguous. The second step was to split PE_r into its GPE_r and IE_r components; the third step was to split IE_r further into its exergy $IE_r - IE_0$ and dead IE_0 components. Finally, the last step was to split APE into its AIE and $AGPE$ components. These successive refinements are illustrated in figure 10 in Appendix B. An important outcome of the analysis is that the structure and form of the $KE/APE/GPE_r$ equations ((1.2)–(1.4)) obtained for the L-Boussinesq model turn out to be more generally valid for a fully compressible thermally stratified fluid, so that one still has

$$\frac{\mathrm{d}\,KE}{\mathrm{d}t} = -C(KE, APE) - D(KE), \tag{2.1}$$

$$\frac{\mathrm{d}\,APE}{\mathrm{d}t} = C(KE, APE) - D(APE), \tag{2.2}$$

$$\frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} = W_{r,mixing} = W_{r,turbulent} + W_{r,laminar}. \tag{2.3}$$

It can be shown, however, that the explicit expressions for $C(KE, APE)$, $D(KE)$ and $D(APE)$ differ between the two sets of equations; see Appendices A and B for the details of these differences. Based on the numerical simulations detailed in the following, the most important point is probably that $D(APE)$ appears to be relatively unaffected by the details of the equation of state, in contrast to $W_{r,mixing}$, which suggests that the L-Boussinesq model is able to accurately represent the irreversible diffusive mixing associated with $D(APE)$. Moreover, since the internal energy contribution to APE is usually small for a nearly incompressible fluid, it also follows that the L-Boussinesq model should also be able to capture the time-averaged properties of $C(KE, APE)$, since the latter is the difference of two terms expected to be accurately represented by the L-Boussinesq model based on the APE equation. The L-Boussinesq model, however, will in general fail to correctly capture the behaviour of GPE_r, unless the approximation of a linear equation of state is accurate enough, as seems to be the case for compositionally stratified flows for instance (e.g. Dalziel *et al.* 2008). The above properties help to rationalize why the L-Boussinesq model appears to perform as well as it often does.

Being reassured that there are no fundamental structural differences between the energetics of the $KE/APE/GPE_r$ system in the Boussinesq and compressible NSE, the next step is to clarify the link with internal energy. One of the main results of this paper, derived in Appendix B, is the following pair of evolution equations for the dead and exergy components of internal energy:

$$\frac{\mathrm{d}\,IE_0}{\mathrm{d}t} \approx D(KE) + D(APE), \tag{2.4}$$

$$\frac{\mathrm{d}\,IE_{exergy}}{\mathrm{d}t} \approx -W_{r,mixing}, \tag{2.5}$$

which were obtained by neglecting terms scaling as $O(\alpha P/(\rho C_p))$, for some values of α, P, ρ and C_p typical of the domain considered, where α is the thermal expansion coefficient, P is the pressure, ρ is the density and C_p is the heat capacity at constant pressure. The important point is that such a parameter is very small for nearly incompressible fluids. For seawater, for instance, typical values encountered in laboratory experiments carried out at atmospheric pressure are $\alpha = 2 \times 10^{-4}\ \mathrm{K}^{-1}$, $P = 10^5\ \mathrm{P_a}$, $C_p = 4 \times 10^3\ \mathrm{J\ K^{-1}kg^{-1}}$, $\rho = 10^3\ \mathrm{m^3kg^{-1}}$, which yield $\alpha P/(\rho C_p) = 5 \times 10^{-6}$. In the deep oceans, this value can increase up to $O(10^{-3})$, but this is still very small. Equations (2.4) and (2.5) confirm that $D(APE)$ and $D(KE)$ are fundamentally similar dissipative processes, in that they both convert APE and KE into dead internal energy, while also confirming that $W_{r,mixing}$ represents a conversion between IE_{exergy} and GPE_r in both the laminar and turbulent cases.

2.2. *Analysis of idealized turbulent mixing events*

To gain insight into the differences between $D(APE)$ and $W_{r,turbulent}$, the energy budget of a hypothetical turbulent mixing event associated with shear flow instability is examined in the light of (2.1)–(2.5). Typically, such events can be assumed to evolve from laminar conditions with no APE. Once the instability is triggered, APE starts to increase and oscillate until the instability subsides and the fluid re-laminarizes, at which point APE returns to zero. The mixing event causes the shear flow to lose a certain amount of kinetic energy $|\Delta KE|$, as well as GPE_r to increase by a certain amount ΔGPE_r, as a result of the partial smoothing out of the mean vertical temperature gradient by molecular diffusion. Integrating (2.1)–(2.5) over the duration of the mixing event yields

$$\Delta KE = -\overline{C(KE,APE)} - \overline{D(KE)}, \tag{2.6}$$

$$\Delta APE = 0 = \overline{C(KE,APE)} - \overline{D(APE)}, \tag{2.7}$$

$$\Delta GPE_r = \overline{W}_{r,turbulent} + \overline{W}_{r,laminar}, \tag{2.8}$$

$$\Delta IE_0 = \overline{D(APE)} + \overline{D(KE)}, \tag{2.9}$$

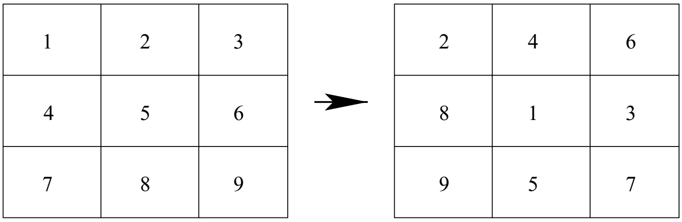
$$\Delta IE_{exergy} = -[\overline{W}_{r,laminar} + \overline{W}_{r,turbulent}], \tag{2.10}$$

where $\Delta(.)$ and the overbar denote a quantity's net change over the time interval and its time-integrated value, respectively. From an observational viewpoint, energy conversion terms such as $\overline{C(KE,APE)}$ are difficult to measure directly; moreover, the results can be ambiguous (e.g. Barry *et al.* 2001 and references therein). As a result, energy conversions are probably best inferred from measuring changes in the different energy reservoirs, as this appears to be easier to do accurately. Multiple possible inferences arise, however, if IE variations are not separated into their IE_{exergy} and IE_0 components. Figure 2 illustrates this point for a hypothetical turbulent mixing experiment with hypothetical plausible numbers, by showing that a given observed net change in IE of $+0.79$ units can potentially be explained – in the absence of any knowledge about the respective variations in IE_0 and IE_{exergy} – as either due to the conversion of 0.8 unit of KE into IE_0 minus the conversion of 0.01 unit of IE_{exergy} into GPE_r or by the conversion of 0.8 unit of KE into IE_0 plus the conversion of 0.2 unit of APE into IE_0 minus the conversion of 0.21 unit of IE_{exergy} into GPE_r. Although the first interpretation is the one implicit in Winters *et al.* (1995) and currently favoured in the literature, it is not possible, based on energy conservation alone, to reject the second interpretation. In fact, the only way to discriminate between the two interpretations requires separately measuring IE_0 and IE_{exergy} variations, as only then are the two interpretations mutually exclusive.

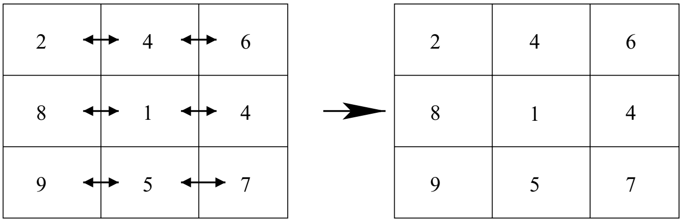
2.3. *An idealized numerical experimental protocol to test the two interpretations*

To compute IE_0 and IE_{exergy}, the knowledge of the temperature field and the Gibbs function for the fluid considered (see Feistel 2003 in the case of water or seawater) is in principle sufficient. It is hoped, therefore, that this study can stimulate laboratory measurements of IE_0 and IE_{exergy}, in order to provide experimental support (or refutation, as the case may be), for the present ideas. In the meantime, numerical methods are probably the only way to assess the accuracy of the two key formulae (2.4) and (2.5), which physically argue (i) that the diffusively dissipated APE is nearly entirely converted into IE_0; and (ii) that GPE_r variations are nearly entirely accounted for by corresponding variations in IE_{exergy}.

To prove our point, energetically consistent idealized mixing events are constructed and studied numerically. The procedure is as follows. One starts from a piece of



(*a*) Stirring



(*b*) Horizontal mixing

FIGURE 3. Idealized depiction of the numerical experimental protocol used to construct figure 4, as well as underlying the method for constructing figures 5 and 6. (*a*) A piece of stratified fluid is cut into pieces of equal mass that are numbered from 1 to N, where N is the total number of parcels. A random permutation is generated as a way to shuffle the parcels randomly and adiabatically, in order to mimic the stirring process. (*b*) All the parcels lying at the same level are homogenized to the same temperature by conserving the total energy of the system, which mimics the horizontal mixing step illustrated in figure 1.

thermally stratified fluid initially lying in its Lorenz (1955)'s reference state in a two-dimensional container with a flat bottom, vertical walls and a free surface exposed at constant atmospheric pressure at its top. The fluid is then discretized on a rectangular array of dimension $N_x \times N_z$ into discrete fluid elements having all the same mass, $\Delta m = \rho \Delta x \Delta z$, where x and z are the horizontal and vertical coordinates respectively, as illustrated in figure 3. The initial stratification has a vertically dependent temperature profile $T(x, P) = T_r(P)$ regarded as a function of the horizontal position x and the pressure P. Thousands of idealized mixing simulations are then generated according to the following procedure:

(*a*) *Initialization of the reference stratification.* The initial stratification is discretized as $T_{i,k} = T(x_i, P_k) = T_r(P_k)$, with $x_i = (i - 1)\Delta x$, $i = 1, \ldots N_x$ and $P_k = P_{min} + (k - 1) g \Delta m$, $k = 1, \ldots N_z$, where P_{min} and $T_r(P_k), k = 1, \ldots N_z$, are randomly generated numbers such that $T_{min} \leqslant T_r(P_k) \leqslant T_{max}$ that have been reordered in the vertical to create a statically stable stratification, for randomly generated T_{min}, T_{max} and P_{min}.

(*b*) *Random stirring of the fluid parcels.* The fluid parcels are then numbered from 1 to $N = N_x \times N_z$ and randomly shuffled by generating a random perturbation of N elements, such that each parcel conserves its entropy in the rearrangement. Such a step is intended to mimic the adiabatic stirring of the parcels associated with the $KE \longrightarrow APE$ conversion. The random stirring of the fluid parcels requires an external amount of energy – called the stirring energy SE – which is diagnosed by computing the difference in potential energy between the shuffled state and the initial state, i.e.

$$SE = (GPE + IE)_{shuffled} - (GPE + IE)_{initial}. \tag{2.11}$$

The latter computation requires knowledge of the thermodynamic properties of the fluid parcels. In this paper, such properties were estimated from the Gibbs function for

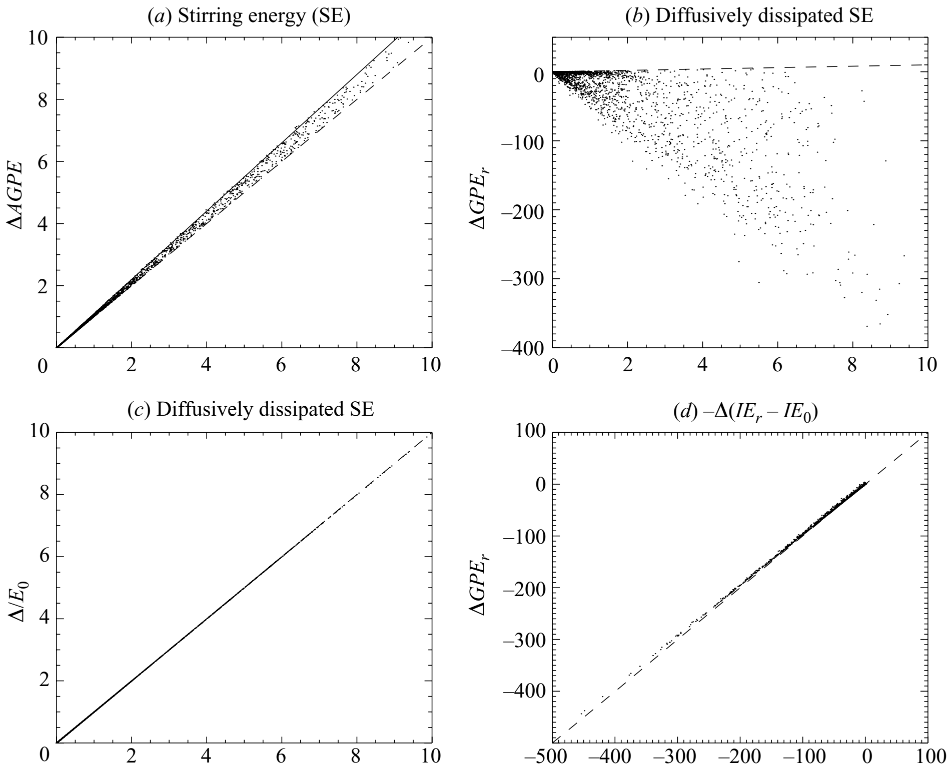


FIGURE 4. (a) The increase of AGPE as a function of the stirring energy SE (see text for details). Each point represents a different stratification shuffled by a different random permutation. The continuous line represents the straight line of equation $\Delta AGPE = SE$, which would describe the energetics of the stirring process if AIE could be neglected. (b) The change of GPE_r as a function of the stirring energy SE dissipated by diffusive mixing. Here, the dotted line is the straight line of equation $\Delta GPE_r = \text{Diffusively dissipated}\, SE$, which would describe the energetics of turbulent mixing if the irreversible conversion $AGPE \longrightarrow GPE_r$ existed. (c) The change in the dead internal energy IE_0 as a function of the diffusively dissipated stirring energy SE. The dashed line is the straight line of equation $\Delta IE_0 = \text{diffusively dissipated}\, SE$. The figure shows a near-perfect correlation. (d) The change in GPE_r as a function of the exergy change. The dashed line is the straight line of equation $\Delta GPE_r = -\Delta IE_{exergy}$. The figure shows, again, a near-perfect correlation.

seawater of Feistel (2003) by specifying a constant value of salinity. Thermodynamic properties such as internal energy, enthalpy, density, entropy, chemical potential, speed of sound, thermal expansion and haline contraction are easily estimated by computing partial derivatives with respect to temperature, pressure, salinity or any combination thereof of the Gibbs function. The stirring energy SE is none other than Lorenz (1955)'s APE of the shuffled state. Since the stirring leaves the background potential energy unaffected, the energetics of the random shuffling is given by

$$\Delta APE = \Delta AGPE + \Delta AIE = SE, \tag{2.12}$$

$$\Delta GPE_r = \Delta IE_r = 0, \tag{2.13}$$

where (2.12) states that the stirring energy SE is entirely converted into APE, while (2.13) expresses the result that being a purely adiabatic process, the stirring leaves the background reference quantities unaltered. Figure 4(a) depicts $\Delta AGPE$ as a function of SE for thousands of experiments, all appearing as one particular point on the plot. According to this figure, $\Delta AGPE$ approximates ΔAPE within about 10%. This illustrates the point that *for adiabatic processes*, APE is well approximated by its gravitational potential energy component as expressed by (1.27).

(*c*) *Isobaric irreversible mixing of the fluid parcels.* In the last step, all the fluid parcels lying in the same layer are mixed uniformly to the same temperature, by assuming an isobaric process that conserves the total enthalpy of each layer. Such a process converts a fraction qSE of the APE into the background PE_r, according to

$$\Delta APE = \Delta AGPE + \Delta AIE = -qSE, \tag{2.14}$$

$$\Delta GPE_r + \Delta IE_r = qSE, \tag{2.15}$$

where $0 < q \leqslant 1$. The factor q is needed here because mixing each layer uniformly does not necessarily lead to a statically stable stratification; when this happens, the resulting stratification still contains some $APE = (1-q)SE$ associated with the static instability, so that $q = 1$ only when the mixed density profile is statically stable. The change in ΔGPE_r resulting from the irreversible mixing step is depicted as a function of the diffusively dissipated stirring energy qSE in figure 4(*b*). If the stirring energy were entirely dissipated into GPE_r, as is classically assumed, then all points should lie on the line of equation $\Delta GPE_r = qSE$ appearing as the dashed line in the figure. Even though such a relation appears to work well in a number of cases, the vast majority of the simulated points corresponds to cases where ΔGPE_r is significantly smaller than qSE, and even often negative as expected in the Fofonoff regime discussed above. On the other hand, if one plots ΔIE_0 as a function of the diffusively dissipated stirring energy qSE, as well as ΔGPE_r as a function of the exergy change $\Delta IE_{exergy} = -\Delta(IE_r - IE_0)$, as done in figure 4(*c*, *d*) respectively, then a visually near-perfect correlation in both cases is obtained. This is consistent with the following relations:

$$\Delta IE_0 \approx qSE, \tag{2.16}$$

$$\Delta GPE_r \approx -\Delta(IE_r - IE_0), \tag{2.17}$$

and hence in agreement with the approximate (2.4) and (2.5). Equation (2.16) empirically verifies (1.28).

2.4. *Numerical estimates of B, $W_{r,mixing}$ and $D(APE)$ as a function of APE*

Having clarified the nature of the net energy conversions occurring in idealized mixing events, we now turn to the estimation of the turbulent rates of the three important conversion terms: B, $W_{r,mixing}$ and $D(APE)$, which are affected by molecular diffusion in the fully compressible Navier–Stokes equations, where B is the work of expansion/contraction. As pointed out in the introduction, enhanced rates fundamentally arise from turbulent fluids possessing large amounts of small-scale APE. For this reason, this paragraph seeks to understand how the values of B, $W_{r,mixing}$ and $D(APE)$ are controlled by the magnitude of APE.

We first focus on the work of expansion/contraction B, which takes the following form:

$$B = \int_V \frac{\alpha P}{\rho C_p} \nabla \cdot (\kappa \rho C_p \nabla T)\, \mathrm{d}V + \int_V \frac{\alpha P}{\rho C_p} \rho \varepsilon \, \mathrm{d}V - \int_V \frac{P}{\rho c_s^2} \frac{DP}{Dt} \, \mathrm{d}V \tag{2.18}$$

obtained by regarding ρ as a function of temperature and pressure. The part of B affected by molecular diffusion is the first term on the right-hand side of (2.18) and is the one under focus here. The second and third terms on the right-hand side are respectively caused by the work of expansion due to the viscous dissipation Joule

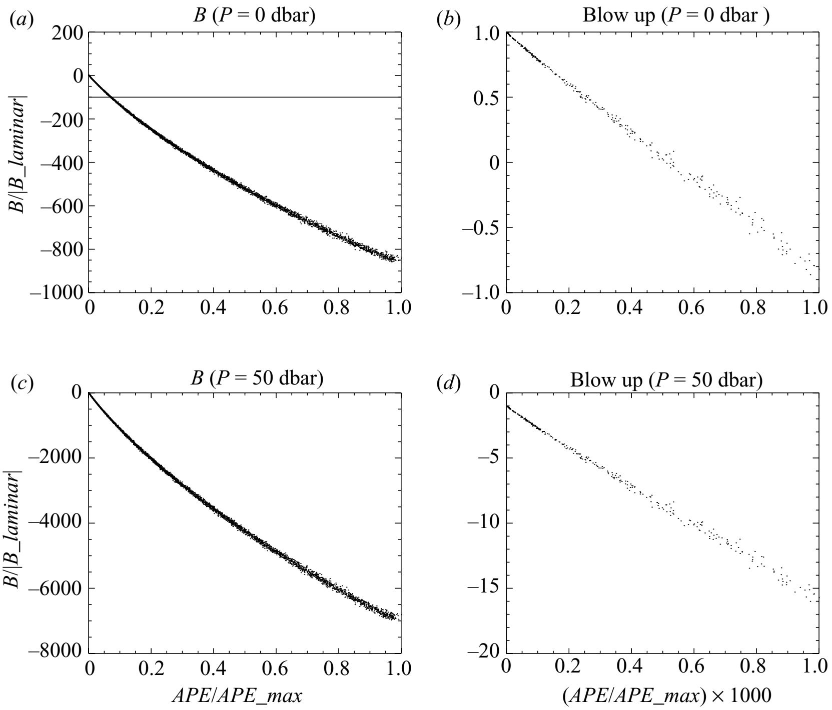


FIGURE 5. (*a, b*) The work of expansion/contraction B normalized by its laminar value (obtained for $APE = 0$) as a function of a normalized APE for a particular stratification corresponding to the classical regime, with (*b*) being a blow-up of (*a*). (*c, d*) Same as the above figure, for the same temperature stratification, but taken at a mean pressure of 50 dbar instead of atmospheric pressure, which is sufficient to put the system in the Fofonoff regime. The figures show that although B is usually negative in every case, it is nevertheless positive for small values of APE in the classical regime, as expected from the L-Boussinesq theory. The normalization constant APE_{max} corresponds to the overall maximum of APE for all experiments.

heating and the adiabatic work of expansion/contraction. The study of these two terms is beyond the scope of this paper.

The diffusive part of B was estimated numerically for thousands of randomly generated stratifications, similar to the previous paragraph, using a standard finite difference discretization of the molecular diffusion operator. Unlike in the previous paragraph, however, all the randomly generated stratifications were computed from only two different reference states pertaining to the classical and Fofonoff regimes respectively, the results are depicted in figure 5(*a–d*) (with figure 5*b,d* providing a blow-up of figure 5*a, c*). The main result is that finite values of APE can make the diffusive part of B negative and considerably larger by several orders of magnitude than in the laminar $APE = 0$ case. This result is important, because it is in stark contrast to what is usually assumed for nearly incompressible fluids at low Mach numbers. From figure 5, it is tempting to conclude that there exists a well-defined relationship between the diffusive part of B and APE, but in fact, the curve $B = B(APE)$ is more likely to represent the maximum value achievable by B for a given value of APE. Indeed, it is important to realize that a given value of APE can correspond to widely different spectral distributions of the temperature field. In the present case, it turns out that the random generator used tend to generate temperature fields with maximum power at small scales, which in turn tend to maximize the value of B for a particular value

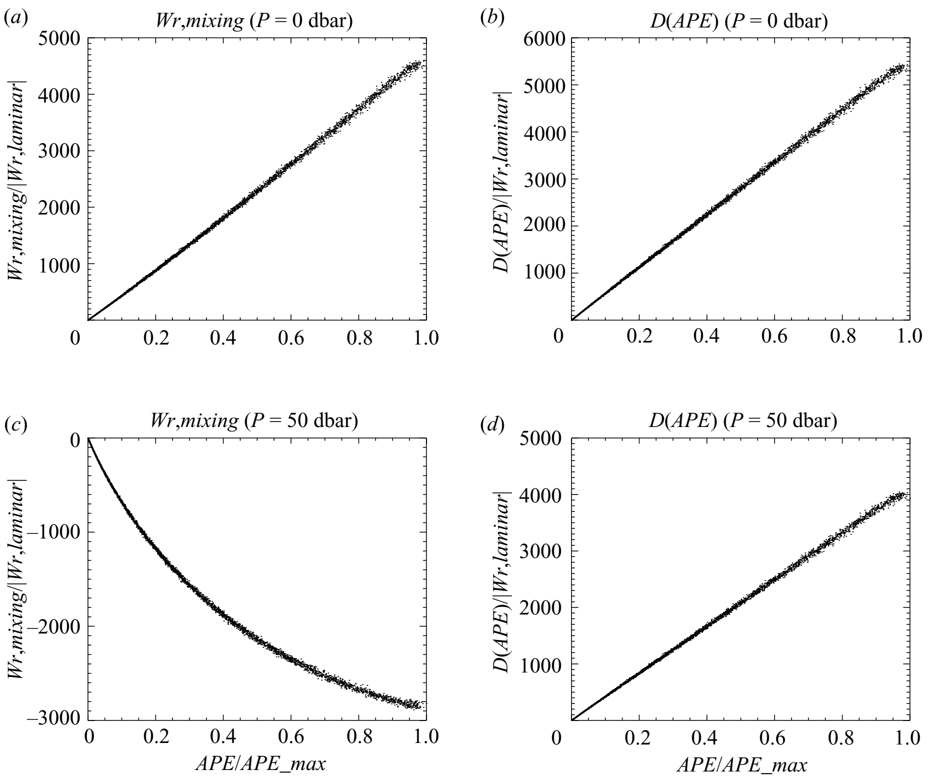


FIGURE 6. (*a, c*) The rate of change of GPE_r normalized by its laminar value as a function of normalized APE, in the classical regime (*a*) as well as in the Fofonoff regime (*c*). The stratification is identical to that of figure 5. (*b, d*) The rate of diffusive dissipation of APE normalized by $W_{r,mixing}$ laminar value, as a function of a normalized APE, in the classical regime (*b*), as well as for the Fofonoff regime (*d*). The figure illustrates the fact that if the former can be regarded as a good proxy for the latter in the classical regime, as is usually assumed, this is clearly not the case in the Fofonoff regime. The two figures also illustrate the fact that the former always underestimates the latter for a thermally stratified fluid, so that observed values of mixing efficiencies obtained from measuring GPE_r variations are necessarily lower bounds for actual mixing efficiencies.

of APE. For the same value of APE, smoother stratifications exist with values of B lying in between the x-axis and the empirical curve $B = B(APE)$, the latter being expected to depend on the numerical grid resolution employed. Nevertheless, figure 5 raises the interesting question of whether the empirical curve $B = B(APE)$ could, in fact, describe the behaviour of the fully developed turbulent regime, an issue that could be explored using direct numerical simulations of turbulence.

The remaining two quantities of interest are $W_{r,mixing}$ and $D(APE)$, which were numerically estimated from the following expressions derived in Appendix B:

$$W_{r,mixing} = \int_V \frac{\alpha_r P_r}{\rho_r C_{pr}} \nabla \cdot (\kappa \rho C_p \nabla T)\, \mathrm{d}V, \tag{2.19}$$

$$D(APE) = \int_V \frac{T_r - T}{T} \nabla \cdot (\kappa \rho C_p \nabla T)\, \mathrm{d}V = \int_V \kappa \rho C_p \nabla T \cdot \left(\frac{T - T_r}{T} \right) \mathrm{d}V. \tag{2.20}$$

As for B, these two quantities were evaluated for thousands of randomly generated stratifications as functions of APE, starting from the same reference states as before. The results for $W_{r,mixing}$ are depicted in figure 6(*a, c*) while the results for $D(APE)$

are depicted in figure 6(*b, d*), with figure 6(*a, b*) and figure 6(*c, d*) corresponding to the classical and Fofonoff regimes respectively. The purpose of the comparison is to demonstrate that whereas there exist stratifications for which the two rates $D(APE)$ and $W_{r,mixing}$ are nearly identical (figure 6*a,b*, the classical regime), as expected from the classical literature about turbulent stratified mixing, it is also easy to construct specific cases occurring in the oceans for which the two rates become of different signs (figure 6*c,d*, the Fofonoff regime). The other important result is the relative insensitivity of $D(APE)$ to the nonlinear character of the equation of state compared with $W_{r,mixing}$, suggesting that the use of the L-Boussinesq model can still accurately describe the KE/APE interactions even for strongly nonlinear equations of states, although it would fail to do a good job of simulating the evolution of GPE_r outside the linear equation of state regime. This also suggests that the L-Boussinesq should be adequate enough to study the mixing efficiency of turbulent mixing events over a wide range of circumstances, provided that by mixing efficiency one means the quantity $\gamma_{mixing} = D(APE)/D(KE)$, and not $\gamma_{mixing} = W_{r,turbulent}/D(KE)$. Finally, we also experimentally verified (not shown) that $D(APE)$ is well approximated by the quantity $W_{r,mixing} - B$, as is expected when AIE is only a small fraction of APE.

2.5. *Synthesis*

The energetics of freely decaying turbulence is summarized in figure 7 for the classical (*a*) and the Fofonoff (*b*) regimes, with figure 7(*c*) attempting further synthesis by combining AIE and $AGPE$ into a single reservoir for APE and the two regimes into a single diagram. Doing so makes figure 7(*c*) basically identical to the Boussinesq energy flowchart depicted in figure 2(*b*). Interestingly, figure 7(*b*) suggests that the Fofonoff regime may differ from the extensively studied classical regime in several fundamental ways. Indeed, whereas both W and B act as net sinks of KE in the classical regime, it appears possible in the Fofonoff regime for some fraction of the KE dissipated into AIE to be recycled back to KE. This is reminiscent of the positive feedback on the turbulent kinetic energy discussed by Fofonoff (1998, 2001), who suggested that such a feedback would enhance turbulent mixing and hence speed up the return to the classical regime after sufficient reduction of the vertical temperature gradient. If real, such a mechanism would be very important to study and understand, as potentially providing a limiting process on the maximum value achievable by the buoyancy frequency, with important implications for numerical ocean models parameterizations. In his papers, however, Fofonoff envisioned the positive feedback on turbulent KE as being associated with the conversion of GPE_r into $AGPE$, but this goes against the findings of this paper arguing that GPE_r can only be exchanged with the exergy reservoir. Fofonoff's feedback mechanism was also criticized by McDougall, Church & Jackett (2003) on different grounds. While the present results do not necessarily rule out Fofonoff's feedback mechanism, they suggest that the latter probably does not work as originally envisioned by Fofonoff, if it works at all (McDougall *et al.* 2003's arguments are not really conclusive either, as they implicitly rely on the existence of the APE/GPE_r conversion). In any case, the issue seems to deserve more attention, given that many places in the oceans appear to fall into Fofonoff's regime.

3. Forced/dissipated balances in the oceans

3.1. *A new approach to the mechanical energy balance in the oceans*

Prior to revisiting MW98's energy constraint (1.34), we start by establishing a number of important results for mechanically and thermodynamically forced thermally

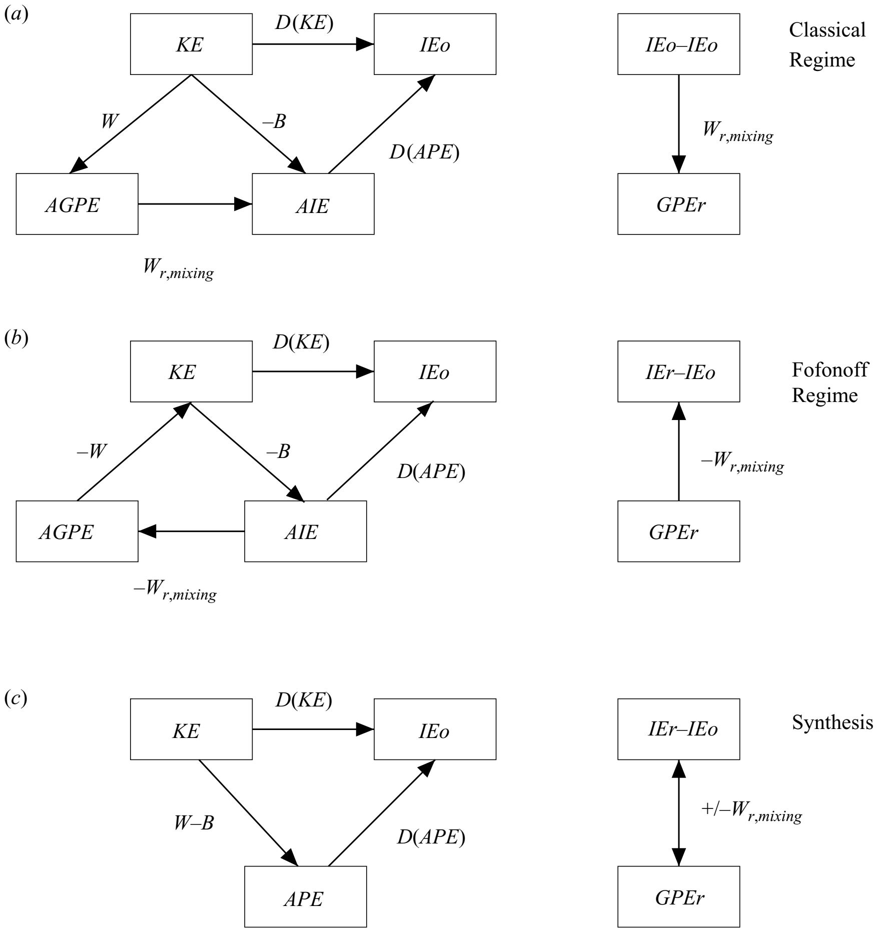


FIGURE 7. The energetics of freely decaying turbulence for the classical regime (*a*), the Fofonoff regime (*b*) and a synthesis of both regimes obtained by subsuming $AGPE$ and AIE into APE alone (*c*). Note the similarity in the energetics in (*c*) and that of the reinterpreted Boussinesq energetics in figure 2(*b*).

stratified fluids, based on the results derived in Appendices A and B. The main modifications brought about by the mechanical and thermodynamical forcing is the apparition of forcing terms, i.e. terms involving the external forcing, in the evolution equations for KE, APE and GPE_r as follows:

$$\frac{\mathrm{d}\,KE}{\mathrm{d}t} = -C(KE, APE) - D(KE) + G(KE), \tag{3.1}$$

$$\frac{\mathrm{d}\,APE}{\mathrm{d}t} = C(KE, APE) - D(APE) + G(APE), \tag{3.2}$$

$$\frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} = \underbrace{W_{r,turbulent} + W_{r,laminar}}_{W_{r,mixing}} - W_{r,forcing}, \tag{3.3}$$

where $G(KE)$ is the work rate done by the external mechanical forcing, $G(APE)$ is the work rate done by the buoyancy forcing and $W_{r,forcing}$ is rate of change of GPE_r (usually a loss, hence the assumed sign convention) due to the buoyancy forcing. The resulting energy transfers are illustrated in the energy flowchart depicted in

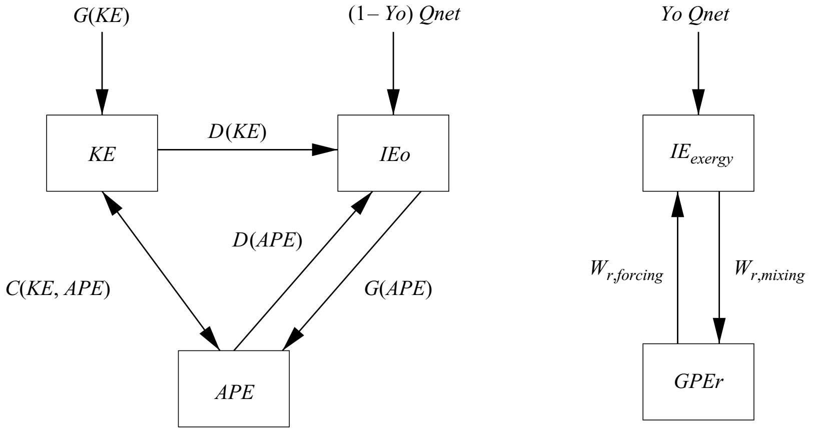


FIGURE 8. Energy flowchart for a mechanically and buoyancy-driven thermally stratified fluid, where $Q_{net} = Q_{heating} - Q_{cooling}$. At leading order, the 'dynamics' (the $KE/APE/IE_0$ system) is decoupled from the 'thermodynamics' (the IE_{exergy}/GPE_r system). The dynamics/thermodynamic coupling occurs due to the correlation between $D(APE)$ and $W_{r,mixing}$, as well as due to the correlation between $G(APE)$ and $W_{r,forcing}$.

figure 8. This figure shows that at leading order, the 'Dynamics' – associated with the reservoirs $KE/APE/IE_0$ – is decoupled from the 'Thermodynamics' – associated with the GPE_r/IE_{exergy} energy reservoirs. Indirect coupling occurs, however, from the fact that $D(APE)$ and $W_{r,turbulent}$ on the one hand, and $G(APE)$ and $W_{r,forcing}$ on the other hand, are strongly correlated with each other.

3.1.1. *Link between $G(APE)$ and $W_{r,forcing}$*

Unlike in the L-Boussinesq model, $G(APE)$ and $W_{r,forcing}$ differ from each other in a real compressible fluid, for the same reasons that $D(APE)$ differs from $W_{r,turbulent}$, as is apparent from their exact formula given by (B 23) and (B 38) in Appendix B:

$$G(APE) = \int_S \frac{T - T_r}{T} \kappa \rho C_p \nabla T \cdot \boldsymbol{n} \mathrm{d}S, \tag{3.4}$$

$$W_{r,forcing} = -\int_S \frac{\alpha_r (P_r - P_a)}{\rho_r C_{pr}} \kappa \rho C_p \nabla T \cdot \boldsymbol{n} \mathrm{d}S. \tag{3.5}$$

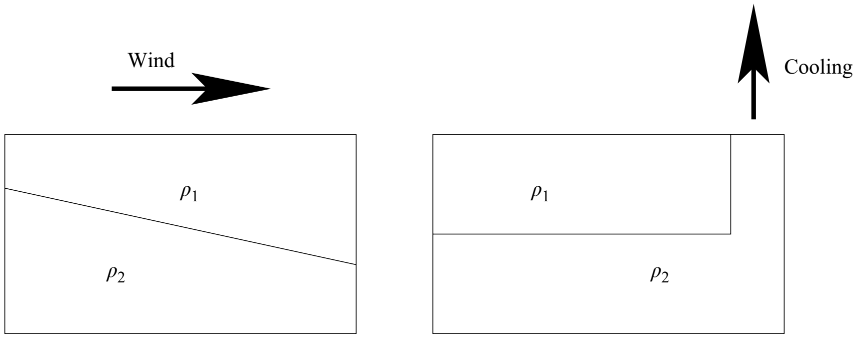
In order to understand by how much $W_{r,forcing}$ differs from $G(APE)$ in a real fluid, it is useful to expand T as a Taylor series around $P = P_r$, i.e.

$$T = T(P_a) = T_r + \Gamma_r (P_a - P_r) + \cdots,$$

where $\Gamma_r = \alpha_r T_r / (\rho_r C_{pr})$ is the adiabatic lapse rate (e.g. Feistel 2003). As a result

$$\frac{T - T_r}{T} = -\frac{\Gamma_r (P_r - P_a)}{T_r} + \cdots \approx -\frac{\alpha_r (P_r - P_a)}{\rho_r C_{pr}}. \tag{3.6}$$

Inserting (3.6) into (3.4) reveals that $G(APE)$ and $W_{r,forcing}$ are in fact equal at leading order. In that case, therefore, the equality between $G(APE)$ and $W_{r,forcing}$ that exactly holds in the L-Boussinesq model appears to be a much better approximation than the corresponding equality between $D(APE)$ and $W_{r,turbulent}$. For this reason, we shall neglect the differences between $G(APE)$ and $W_{r,forcing}$ in the following section.



(*a*) Production of APE by wind forcing conversion *C*(*KE*, *APE*) > 0

(*b*) Production of APE by surface cooling conversion *C*(*APE*, *KE*) > 0

FIGURE 9. Idealized depictions of mechanically driven (*a*) and buoyancy-driven (*b*) creation of APE. (*a*) A wind blowing at the surface of a two-layer fluid causes the tilt of the layer interface, resulting in a net $C(KE, APE) > 0$ conversion. (*b*) Localized cooling at high latitudes sets the density of a fraction of the upper layer to that of the bottom layer, also inducing a tilt in the layer interface. The return of the interface to equilibrium conditions (flat interface) results in a net $C(APE, KE) > 0$ conversion.

3.1.2. *Steady-state mechanical energy balance*

Under steady-state conditions, summing (3.1) and (3.2) yields

$$G(KE) + G(APE) = D(APE) + D(KE), \tag{3.7}$$

which simply states that in a steady state, the production of mechanical energy by the wind and buoyancy forcing is balanced by the viscous and diffusive dissipations of KE and APE, respectively. Figure 9 schematically illustrates how both the wind and the buoyancy forcing can contribute to the creation of APE. The main novelty here is to make it clear that $D(APE)$ is a ‘true’ dissipation mechanism, i.e. one that degrades mechanical energy into internal energy (as does viscous dissipation), not one converting mechanical energy into another form of mechanical energy (i.e. GPE_r). This suggests regarding $D(APE)+D(KE)$ as the total dissipation of available mechanical energy, $ME = KE + APE$.

In contrast, most studies of ocean energetics of the past decade have tended to subsume the APE production and dissipation terms into the single term $B = G(APE) - D(APE)$, in which case (3.7) becomes

$$G(KE) + B = D(KE). \tag{3.8}$$

The problem in writing the mechanical energy balance under this form is that it erroneously suggests that B, rather than $G(APE)$, is the work rate done by surface buoyancy fluxes, and that viscous dissipation is the only form of mechanical energy dissipation. For instance, Wang & Huang (2005) estimated $B = O(1.5\,\mathrm{GW})$ in the oceans, in the context of the L-Boussinesq model, which is about three orders of magnitude less than the work rate done by the wind and tides. In one of the most recent reviews about ocean energetics by Kuhlbrodt *et al.* (2007), it is Wang & Huang (2005)’s estimate for B that is presented as the work rate done by surface buoyancy fluxes, while Oort *et al.* (1994)’s previous result for $G(APE)$ is regrettably omitted.

In the L-Boussinesq model, B takes the particular form $B = \kappa g(\langle P\rangle_{bottom} - \langle P\rangle_{top})$, where $\langle P\rangle_{top}$ and $\langle P\rangle_{bottom}$ are the area-integrated surface and bottom density

R_a	10^6	2×10^6	3×10^6	4×10^6	5×10^6
Φ	6.2	6.9	7.5	7.9	8.3

TABLE 1. Values of Φ as a function of the Rayleigh number R_a reproduced from figure 4 of Paparella & Young (2002). Values of R_a appropriate to the oceans are of the order $R_a = O(10^{20})$.

respectively, so that in the absence of mechanical forcing, (3.8) becomes

$$\kappa g(\langle P\rangle_{bottom} - \langle P\rangle_{top}) = D(KE). \tag{3.9}$$

In a study addressing the issue of horizontal convection, recently reviewed by Hughes & Griffiths (2008), Paparella & Young (2002) proved that the left-hand side of (3.9) must be bounded by κ times some finite constant when the fluid is forced by a surface temperature condition, with no-normal flux applying everywhere else. This result is now commonly referred to as the 'anti-turbulence theorem', for the bound implies that $D(KE)$ must vanish in the 'inviscid' limit (used here to mean both vanishing molecular viscosity *and* diffusivity), thus violating the so-called zeroth law of turbulence, an empirical law grounded in many observations showing that the viscous dissipation of KE in homogeneous turbulent fluid flows remains finite and independent of molecular viscosity as the Reynolds number is increased indefinitely.

3.1.3. *Actual implications of the anti-turbulence theorem*

As shown by Wang & Huang (2005), Paparella & Young (2002)'s bound suggests that the oceanic viscous dissipation $D(KE)$ would be less than 1.5 GW in the absence of mechanical forcing. Since this value is several orders of magnitude than observed oceanic values of $D(KE)$, the result demonstrates that mechanical forcing is essential to account for the latter. By itself, however, the result says nothing about whether mechanical forcing is also essential to account for the observed turbulent rates of diapycnal mixing, since, as far we are aware, the values of R_f and γ_{mixing} for horizontal convection have never been determined before. Indeed, the anti-turbulence theorem only imposes that the difference $B = G(APE) - D(APE)$ be small, but this does not forbid $G(APE)$ and $D(APE)$ to be individually very large. In fact, the current theoretical and numerical evidence suggests that $G(APE)$ and $D(APE)$ increase with the Rayleigh number R_a. Indeed, this is suggested by Paparella & Young (2002)'s numerical experiments, which show the function Φ (given by (1.1)), which we interpreted as a measure of the Cox number $O(K_T/\kappa)$ (and hence of $D(APE)$) to increase with $R_a = g\alpha\Delta T H^3/(\nu\kappa)$ as tabulated in table 1. Although such values appear to be much smaller than observed $O(10^2\text{–}10^3)$ Cox numbers, they also correspond to Rayleigh numbers that are about 13–14 orders of magnitude smaller than occurring in the oceans, leaving open the possibility for Φ to be possibly much larger, possibly as large as encountered in the oceans. In a related study, Siggers, Kerswell & Balmforth (2004) derived a bound on Φ (which they related to a horizontal Nusselt number), which does not exclude the possibility that horizontal convection, on its own, could support a north-south heat transport of the observed magnitude. A further discussion of the physics of horizontal convection based on laboratory experiments is provided by Mullarney, Griffiths & Hughes (2004). In summary, while the anti-turbulence theorem demonstrates the need for mechanical forcing to account for the observed values of kinetic energy dissipation, the question of whether mechanical forcing is needed to sustain diapycnal mixing rates and a north-south heat transport of the observed magnitude is still largely open. In particular, it is important to point out that although the anti-turbulence theorem rules out the possibility of elevated values of kinetic energy dissipation in the absence of mechanical forcing, it does not rule out the possibility of elevated values of diapycnal mixing rates. In this respect, Paparella & Young (2002)'s suggestion that horizontal convection should be regarded as 'non-turbulent' appears somewhat misleading.

3.1.4. *Back-of-the-envelope estimate of $G(APE)$ for the world oceans*

In order to make progress on the above issue, it is essential to determine how large $G(APE)$ can be in the oceans. As mentioned above, Oort *et al.* (1994) inferred $G(APE) = 1.2 \pm 0.7\,\mathrm{TW}$ from observations, and hence to be nearly as large as the work rate done by the wind, but this estimate was questioned by MW98 on the basis of Sandstrom's 'theorem'. A possible source of error in Oort *et al.* (1994)'s is its reliance on the so-called Lorenz approximation, which is often said to overestimate $G(APE)$ (e.g. Huang 1998).

In fact, the simplest method to convince oneself that $G(APE)$ must be large in the oceans comes from the result that $G(APE) \approx W_{r,forcing}$ established previously, which states that if the rate of decrease of GPE_r due to the buoyancy forcing is large, so must it be the case for $G(APE)$. Since MW98 inferred $W_{r,forcing} \approx 0.4\,\mathrm{TW}$, one can immediately conclude $G(APE) \approx 0.4\,\mathrm{TW}$, which, it turns out, is close to the lower bound of Oort *et al.* (1994)'s estimate, consistent with the idea that the method used by the latter should overestimate $G(APE)$. This immediately establishes that MW98's assumption that $G(APE)$ is small is inconsistent with their assumption that $W_{r,forcing}$ is large. It also establishes that Sandstrom's 'theorem', whatever it means, cannot say anything meaningful about $G(APE)$.

An independent way to estimate $G(APE)$ is by using the exact formula for $G(APE)$ or $W_{r,forcing}$ given by (3.4) and (3.5) recalled above. In these formulae, T_r is the temperature that a surface parcel would have if lifted adiabatically to its reference level. Since the oceans are on average heated and cooled where they are the warmest and coolest respectively, the parcels' reference level will be on average close to the surface in the warm regions, but much deeper in the cold regions. Equation (3.4) must therefore be dominated by surface cooling. Using the near equality between $G(APE)$ and $W_{r,forcing}$, we take as our estimate for $G(APE)$ the expression:

$$G(APE) \approx \left(\frac{\alpha_r P_r}{\rho_r C_{pr}}\right)_{cooling} Q_{cooling} \tag{3.10}$$

with a value of $(\alpha_r P_r/(\rho_r C_{pr}))_{cooling}$ appropriate to the regions of cooling. Using the values $\alpha = 5 \times 10^{-5}, \mathrm{K}^{-1}$, $P_r = 2000\,\mathrm{dbar} = 2 \times 10^7\,\mathrm{Pa}$, $\rho_r = 10^3\,\mathrm{kg}$, $C_{pr} = 4 \times 10^3\,\mathrm{J\ kg^{-1}K^{-1}}$, and $Q_{cooling} = 2\,\mathrm{PW}$ yields

$$G(APE) = \frac{5 \times 10^{-5} \times 2 \times 10^7}{10^3 \times 4 \times 10^3} \times 2 \times 10^{15}\,\mathrm{W} = 0.5\,\mathrm{TW},$$

which is very close to Munk & Wunsch (1998)'s estimate for $W_{r,forcing}$ and is consistent with Oort *et al.* (1994)'s lower bound for $G(APE)$. The large value of $G(APE)$ suggests that buoyancy forcing can actively participate in maintaining turbulent diapycnal mixing in the oceans, which should be reflected by a higher value of γ_{mixing} than the value $\gamma_{mixing} = 0.2$ currently used in the literature about the subject.

3.2. *A new look at the GPE_r balance and Munk & Wunsch (1998)'s theory*

Having clarified the 'available mechanical energy balance', we now turn to the GPE_r budget, with the aim of elucidating the assumptions underlying MW98's constraint on the energy requirement for sustaining diapycnal mixing in the oceans. In a steady state, the GPE_r budget given by (3.3) becomes

$$W_{r,mixing} = W_{r,turbulent} + W_{r,laminar} = W_{r,forcing}, \tag{3.11}$$

where the explicit expressions for $W_{r,mixing}$ and $W_{r,forcing}$ are given at leading order by the following expressions:

$$W_{r,mixing} \approx \int_V \kappa \rho C_p \nabla T \cdot \nabla \left(\frac{\alpha_r T_r}{\rho_r C_{pr}} \right) \mathrm{d}V, \tag{3.12}$$

$$W_{r,forcing} \approx \int_S \left(\frac{\alpha_r P_r}{\rho_r C_{pr}} \right) \kappa \rho C_p \nabla T \cdot \boldsymbol{n} \mathrm{d}S, \tag{3.13}$$

which are valid for a fully compressible thermally stratified ocean where $\boldsymbol{n}$ is the unit vector normal to the surface. Note that for the L-Boussinesq model, implicitly considered by MW98, $W_{r,mixing}$ can be rewritten as

$$W_{r,mixing} \approx \int_V \kappa \|\nabla z_r\|^2 \alpha_r \frac{\partial T_r}{\partial z_r} \mathrm{d}V \approx \int_V K_T \rho_0 N_r^2 \mathrm{d}V \tag{3.14}$$

by using the definition of turbulent diapycnal diffusivity of Winters *et al.* (1995) for K_T. As (3.14) is exactly the expression used by MW98, this establishes that MW98's analysis actually pertains to the GPE_r budget, not the GPE budget, and that their results should logically follow from (3.11).

To show that this is indeed the case, simply use the definition $\gamma_{mixing} = D(APE)/D(KE)$ in combination with the mechanical energy balance to express $D(APE)$ in terms of the total mechanical energy input $G(APE) + G(KE)$ as follows:

$$D(APE) = \frac{\gamma_{mixing}}{1 + \gamma_{mixing}} [G(APE) + G(KE)] = R_f [G(APE) + G(KE)], \tag{3.15}$$

where $R_f = \gamma_{mixing}/(1 + \gamma_{mixing})$ is the dissipation flux Richardson number defined in the introduction. Now, writing $W_{r,turbulent} = \xi D(APE)$ as proposed in this paper to account for a nonlinear equation of state, neglecting $W_{r,laminar}$ compared with $W_{r,turbulent}$ and using the result that $W_{r,forcing} \approx G(APE)$ demonstrated previously, (3.11) becomes

$$\xi D(APE) \approx G(APE). \tag{3.16}$$

The desired result is obtained by combining (3.15) and (3.16), which yields

$$G(KE) \approx \frac{1 + (1 - \xi)\gamma_{mixing}}{\xi \gamma_{mixing}} G(APE) = \frac{1 - \xi R_f}{\xi R_f} G(APE). \tag{3.17}$$

This formula generalizes MW98's equation (1.34) to account for a nonlinear equation of state, the effects of which are contained in the single parameter $\xi < 1$. It is easily seen that MW98's equation (1.34) is recovered simply by setting $\xi = 1$ in (3.17), using the result that $G(APE) \approx W_{r,forcing}$. This formula is further extended and discussed by Tailleux & Rouleau (2009), in the context of idealized experiments of mechanically stirred horizontal convection.

The confirmation that $G(APE)$ should actually be nearly as large as the work rate done by the mechanical forcing makes it possible for buoyancy forcing to drive possibly a very large fraction of the oceanic turbulent diapycnal mixing, which should

	$\gamma_{mixing}=0.2$	$\gamma_{mixing}=0.5$	$\gamma_{mixing}=1$
$\xi=1$	2 TW	0.8 TW	0.4 TW
$\xi=0.5$	4.4 TW	2 TW	1.2 TW

TABLE 2. Mechanical energy requirements on $G(KE)$ depending on different assumed values for the dissipations ratio γ_{mixing} and nonlinearity parameter ξ, as computed from (3.17) using $G(APE)=0.4$ TW, in line with Munk & Wunsch (1998)'s assumptions.

be reflected in an appropriate value for γ_{mixing}. As noted in the introduction, buoyancy-driven turbulent mixing is usually significantly more efficient than mechanically driven turbulent mixing, suggesting that a value of γ_{mixing} significantly larger than the value $\gamma_{mixing}=0.2$ should be used in (3.17). Likewise, the fact that the nonlinear equation of state for seawater is found to strongly affect changes in GPE_r, as shown in the previous section, suggests that a value of ξ between 0 and 1 should be used (note that ξ cannot be negative if a steady state is to exist). A detailed discussion of which values should actually be used for γ_{mixing} and ξ is beyond the scope of this paper, however, as much more needs to be understood about buoyancy-driven and mechanically driven turbulent mixing in a non-Boussinesq stratified fluid before one may become confident enough to speculate on the 'right' values. In order to fix ideas, however, it is useful to compute the energy requirement on turbulent mixing predicted by (3.17) for plausible values of ξ and γ_{mixing}, as reported in table 2, under the assumption that $W_{r,forcing} \approx G(APE) \approx 0.4$ TW.

As expected, decreasing ξ at fixed γ_{mixing} increases the requirement on $G(KE)$. This is consistent with Gnanadesikan *et al.* (2005)'s conclusions that cabelling (i.e. the contraction upon mixing stemming from the nonlinear character of the equation of state for seawater) increases the requirement on $G(KE)$. Likewise, increasing γ_{mixing} at fixed ξ decreases the requirement on $G(KE)$. This is consistent with Hughes & Griffiths (2006)'s argument that the requirement on $G(KE)$ can be decreased if the entrainment of ambient water by the sinking cold plumes is accounted for. Indeed, taking into account entrainment effects is equivalent to increasing γ_{mixing}, since entrainment is physically associated with buoyancy-driven turbulent mixing, as far as we understand the issue. Note that decreasing ξ seems to require increasing γ_{mixing} if one accepts the idea that no more than 2 TW is available from mechanical energy sources to stir the oceans. Alternatively, one could also perhaps question the assumption that the oceans are truly in a steady state.

4. Summary and conclusions

In this paper, we extended the APE framework of Winters *et al.* (1995) to the fully compressible Navier–Stokes equations, with the aims of clarifying (i) the nature of the energy conversion taking place in turbulent stratified fluids; and (ii) the role of the surface buoyancy fluxes in Munk & Wunsch (1998)'s constraint on the mechanical sources of stirring required to sustain diapycnal mixing in the oceans. The most important results are that the well-known turbulent increase in background GPE_r, commonly thought to occur at the expense of the diffusively dissipated APE, actually occurs at the expense of (the exergy part of) IE. On the other hand, the APE dissipated by molecular diffusion is found to be dissipated into (the dead part of) IE, i.e. the same kind of IE the viscously dissipated KE is converted into, not into GPE_r. Turbulent stirring, therefore, should not be viewed as introducing a new form of mechanical-to-mechanical APE/GPE_r conversion, but simply as enhancing the existing IE/GPE_r conversion rate, in addition to enhancing the viscous dissipation rate of KE, as well as the diffusive entropy production and APE dissipation rates. These results are important, for they significantly alter the current understanding on the nature of turbulent diapycnal mixing and its links with the dissipation of mechanical energy and turbulent increase of GPE_r. In particular, the possibility that GPE_r may decrease as a result of turbulent mixing, not necessarily increase as is commonly thought, is to be emphasized. Moreover, the fact that the turbulent increase of GPE_r is associated with an enhanced IE/GPE_r conversion physically implies that compressible effects must be considerably larger than previously thought, raising fundamental questions about the possible limitations of the widely used incompressible assumption in the modelling of fluid flows at low Mach numbers, which further work should elucidate. Finally, the present results also have implications for the way one should quantify the efficiency of mixing in turbulent stratified fluids, with new definitions for the mixing efficiency γ_{mixing} (or more accurately dissipations ratio) and the flux Richardson number R_f being proposed in the introduction, where they are also compared with existing definitions.

A significant achievement of the extended APE framework is to allow for a more rigorous and general rederivation of MW98's result (our (3.17)), which is also valid for a non-Boussinesq ocean and results in the appearance of a nonlinearity parameter $\xi \leqslant 1$, with MW98's results being recovered for $\xi = 1$. The main new result here is the finding that the work rate done by the surface buoyancy fluxes $G(APE)$ should be numerically comparable with $W_{r,forcing}$. This is important because, while $W_{r,forcing}$ is currently widely agreed to be large $O(0.4\,\mathrm{TW})$, $G(APE)$ has been widely thought to be negligible on the basis of MW98's argument that this is required by Sandstrom's 'theorem'. The result therefore demonstrates that $G(APE)$ is as important as the mechanical forcing in driving and stirring the oceans, in agreement with Oort *et al.* (1994)'s previous conclusions. The two main consequences are that: (i) there is no reason to reject the idea that the oceans are a heat engine and that the north-south heat transport is mostly the result of the buoyancy forcing, usually thought to be the case prior to MW98's study (e.g. Colin de Verdière 1993); and (ii) the overall value of γ_{mixing} in the oceans is likely to be significantly larger than the value of $\gamma_{mixing} = 0.2$ currently used, as seems to be required by the large magnitude of $G(APE)$, given that buoyancy-driven turbulent mixing has a significantly higher γ_{mixing} than mechanically driven turbulent mixing in general. Note that increasing γ_{mixing} decreases the requirement on the mechanical sources of stirring, thereby providing a natural way to remove the apparent paradoxical result of a shortfall in the mechanical stirring energy that arises when $\gamma_{mixing} = 0.2$ is used in (3.17) for $\xi = 1$. The extended APE framework may also be used to revisit the assumed implications of Paparella & Young (2002)'s anti-turbulence 'theorem', the main conclusion being that horizontal convection may, in fact, support elevated values of diapycnal mixing, in contrast to what is usually assumed. It seems important to stress, however, that even if one could prove that surface buoyancy fluxes could sustain on their own diapycnal mixing rates as large as observed, it would not prove that the actual buoyancy-driven component of the ocean circulation is not mechanically controlled. Indeed, it is essential to recognize that both $G(KE)$ and $G(APE)$ depend not only on the external forcing parameters but also strongly on the actual ocean circulation and stratification, which necessarily implies a mechanical control of the buoyancy-driven circulation and vice versa. In other words, even though the present study disagrees with MW98's contention that surface buoyancy fluxes do not work or stir the oceans, there should be little doubt that even if the MOC is expected to be primarily buoyancy-driven, it is also mechanically controlled in ways that are not fully understood. In this respect, therefore, MW98 deserves credit for challenging the idea that the behaviour of the buoyancy-driven circulation can be understood independently of its link with the mechanical forcing.

An important challenge ahead is to extend the present framework for dealing with an equation of state that depends on salinity, which gives rise to the possibility of double-diffusion effects and storing energy in chemical form. Such extensions are needed to better connect the present results to many laboratory experiments based on the use of compositionally stratified fluids, although the present evidence is that the use of a linear equation of state is probably accurate enough to describe the latter. To that end, many technical difficulties need to be overcome. Indeed, salinity complicates the definition of Lorenz (1955)'s reference state to such an extent that it is not even clear that such a state can be uniquely defined (e.g. Huang 2005), as may be the case in presence of humidity in the atmosphere (e.g. Tailleux & Grandpeix 2004). A potentially important generalization would also be to further decompose the internal energy in order to isolate the available acoustic energy considered by Bannon (2004), which in the present paper is included as part of our definition of APE. Finally, more work is required to understand how to fix the values of ξ and γ_{mixing} in (3.17) in the actual oceans.

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Appendix A. Energetics of Incompressible Navier–Stokes Equations

A.1. *Boussinesq equations with equation of state nonlinear in temperature*

This appendix documents the energetics of the Boussinesq system of equations that form the basis for most inferences about stratified turbulence for fluid flows at low Mach numbers and which is commonly used in the theoretical and numerical study of turbulence (e.g. Winters *et al.* 1995; Caulfield & Peltier 2000; Staquet 2000; Peltier & Caulfield 2003). In order to go beyond the usual case of a linear equation of state, a slight generalization is introduced by allowing the thermal expansion coefficient to vary with temperature. The resulting set of equations is therefore as follows:

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \frac{1}{\rho_0}\nabla P = -\frac{g\rho}{\rho_0}\hat{\boldsymbol{z}} + \nu\nabla^2\boldsymbol{v}, \tag{A 1}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{A 2}$$

$$\frac{\mathrm{D}T}{\mathrm{D}t} = \kappa \nabla^2 T, \tag{A 3}$$

$$\rho(T) = \rho_0 \left[1 - \int_{T_0}^{T} \alpha(T')\mathrm{d}T' \right], \tag{A 4}$$

where $\boldsymbol{v} = (u, v, w)$ is the three-dimensional velocity field, P is the pressure, ρ is the density, T is the temperature, $\nu = \mu/\rho$ is the kinematic viscosity, μ is the (dynamic) viscosity, κ is the molecular diffusivity, g is the acceleration of gravity and ρ_0 is a reference density. The classical Boussinesq model, called the L-Boussinesq model in this paper, is simply recovered by taking α to be a constant in (A 4). In that case, (A 3) and (A 4) may be combined to obtain the following diffusive model for density:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \kappa \nabla^2 \rho \tag{A 5}$$

as assumed in many numerical studies of turbulence (e.g. Winters *et al.* 1995; Caulfield & Peltier 2000; Staquet 2000; Peltier & Caulfield 2003). When the temperature dependence of α is retained, the resulting model is called the NL-Boussinesq model.

A.2. *Standard energetics*

Evolution equations for the KE and GPE are obtained by the standard procedure (e.g. Batchelor 1967; Landau & Lifshitz 1987), assuming that the system is forced mechanically by an external stress τ, and thermodynamically by external heat fluxes, both assumed to act at the surface boundary located at $z = 0$. The first equation is premultiplied by $\rho_0 \boldsymbol{v}$ and volume-integrated. After reorganization, the equation becomes

$$\frac{\mathrm{d}\,KE}{\mathrm{d}t} = \frac{d}{\mathrm{d}t}\int_V \rho_0 \frac{\boldsymbol{v}^2}{2}\mathrm{d}V = \underbrace{\int_{\partial V} \tau \cdot \boldsymbol{u}_s \mathrm{d}S}_{G(KE)} - \underbrace{\int_V \rho g w \,\mathrm{d}V}_{W} + \underbrace{\int_V \rho_0 \varepsilon \,\mathrm{d}V}_{D(KE)}, \tag{A 6}$$

where W is the so-called density flux, $D(KE)$ is the viscous dissipation rate of kinetic energy and $G(KE)$ is the rate of work done by the external stress. The time evolution of the total gravitational potential energy of the fluid, i.e. the volume integral of $\rho g z$, is

$$\frac{\mathrm{d}\,GPE}{\mathrm{d}t} = \frac{d}{\mathrm{d}t}\int_V \rho g z \,\mathrm{d}V = \underbrace{\int_V \rho g w \,\mathrm{d}V}_{W} \underbrace{- \int_V \rho_0 g z \alpha \kappa \nabla^2 T \,\mathrm{d}V}_{B}, \tag{A 7}$$

where B is the Boussinesq approximation of the work of expansion/contraction. In the present case, it is possible to derive an explicit analytical formula for B:

$$\begin{aligned} B &= -\int_V g z \rho_0 \alpha \kappa \nabla^2 T \,\mathrm{d}V = \int_V \kappa g \rho_0 \alpha \frac{\partial T}{\partial z}\,\mathrm{d}V + \int_V \kappa \rho_0 g z \frac{\mathrm{d}\alpha}{\mathrm{d}T}(T) \|\nabla T\|^2 \,\mathrm{d}V \\ &= \underbrace{\kappa g [\langle \rho \rangle_{bottom} - \langle \rho \rangle_{top}]}_{B_L} + \int_V \kappa \rho_0 g z \frac{\mathrm{d}\alpha}{\mathrm{d}T}(T) \|\nabla T\|^2 \,\mathrm{d}V, \end{aligned} \tag{A 8}$$

by using integration by parts, and using the fact that the surface term vanishes because the surface is by assumption located at $z = 0$, where $\langle \rho \rangle_{bottom}$ and $\langle \rho \rangle_{top}$ denote the surface-integral of the bottom and top values of density. For a linear equation of state, $B = B_L$ will, in general, be small because of the smallness of the molecular diffusivity κ, and finite top–bottom density difference. When α increases with temperature, however, B may become significantly larger than B_L in turbulence strong enough to make $\|\nabla T\|^2$ large for the second term in (A 8) to overcome B_L, pointing out the possible critical role of nonlinearity of the equation of state in strongly turbulent fluids.

A.3. *Lorenz (1955)'s available energetics*

We now seek evolution equations for the available and unavailable parts of the gravitational potential energy, as previously done by Winters *et al.* (1995) in the case of the L-Boussinesq equations. By definition, the expression for the GPE_r is

$$GPE_r = \int_V \rho_r g z_r \, \mathrm{d}V, \tag{A 9}$$

where $z_r = z_r(\boldsymbol{x}, t)$ and $\rho_r = \rho_r(z_r, t)$ are the vertical position and the density of the parcels in the Lorenz (1955)'s reference state. In Boussinesq models, fluid parcels are assumed to conserve their *in situ* temperature in the reference state, so that $T_r(z_r, t) = T(\boldsymbol{x}, t)$. Taking the time derivative of (A 9) yields

$$\begin{aligned}\frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} &= \int_V g z_r \frac{\mathrm{D}\rho_r}{\mathrm{D}t}\mathrm{d}V + \underbrace{\int_V g\rho_r \frac{\mathrm{D}z_r}{\mathrm{D}t}\mathrm{d}V}_{=0} = -\int_V g z_r \rho_0 \alpha\kappa \nabla^2 T \,\mathrm{d}V \\ &= -\int_{\partial V} g z_r \rho_0 \alpha \kappa \nabla T \cdot \boldsymbol{n}\,\mathrm{d}S + \int_V \kappa \rho_0 g \nabla T \cdot \nabla(\alpha z_r)\mathrm{d}V \\ &= -W_{r,forcing} + \underbrace{\int_V \kappa\rho_0 g \|\nabla z_r\|^2 \frac{\partial(\alpha z_r)}{\partial z_r}\frac{\partial T_r}{\partial z_r}\mathrm{d}V}_{W_{r,mixing}},\end{aligned} \tag{A 10}$$

where $W_{r,forcing}$ is the rate of change of GPE_r due to the external surface heating/cooling. For a Boussinesq fluid, this term is identical to the APE production rate $G(APE)$, as shown in the following, i.e. $W_{r,forcing} = G(APE)$. The above formula was obtained by using the following intermediate results:

$$\nabla[T(\boldsymbol{x})] = \nabla[T_r(z_r(\boldsymbol{x}))] = \frac{\partial T_r}{\partial z_r}\nabla z_r, \tag{A 11}$$

$$\nabla[\alpha(T) z_r] = \nabla[\alpha(T_r(z_r)) z_r] = \frac{\partial(\alpha z_r)}{\partial z_r}\nabla z_r, \tag{A 12}$$

as well as the important result that the integral involving the term $\mathrm{D}z_r/\mathrm{D}t$ vanishes identically established by Winters *et al.* (1995). In this paper, the result was established by using an explicit formula for the reference stratification. An alternative way to recover such a result is achieved by noting that the velocity $\boldsymbol{v}_r = (\mathrm{D}x_r/\mathrm{D}t, \mathrm{D}z_r/\mathrm{D}t)$ of the fluid parcels in the reference state must satisfy the continuity equation

$$\nabla_r \cdot \boldsymbol{v}_r = 0, \tag{A 13}$$

where ∇_r is the divergence operator in the reference space state (x_r, y_r, z_r), from which it follows that the surface integral of $W_{r,mixing} = \mathrm{D}z_r/\mathrm{D}t$ along each constant z_r level must vanish, which implies Winters *et al.* (1995)'s result. The equation for $APE = AGPE = GPE - GPE_r$ becomes

$$\begin{aligned}\frac{\mathrm{d}\,APE}{\mathrm{d}t} &= \frac{\mathrm{d}\,GPE}{\mathrm{d}t} - \frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} \\ &= W - (W_{r,mixing} - B) + W_{r,forcing} = G(APE) + W - D(APE),\end{aligned} \tag{A 14}$$

where

$$D(APE) = W_{r,mixing} - B$$
$$= \underbrace{\int_V \rho_0 g\alpha\kappa \|\nabla z_r\|^2 \frac{\partial T_r}{\partial z_r} \mathrm{d}V - B_L}_{D_L(APE)} + \underbrace{\int_V \kappa\rho_0 g(z_r - z)\frac{\mathrm{d}\alpha_r}{\mathrm{d}T_r}\|\nabla z_r\|^2 \left(\frac{\partial T_r}{\partial z_r}\right)^2 \mathrm{d}V}_{D_{NL}(APE)}, \tag{A 15}$$

by using the results that

$$\int_V \rho_0 g\kappa \frac{\mathrm{d}\alpha}{\mathrm{d}T}\|\nabla T\|^2 \mathrm{d}V = \int_V \rho_0 g\kappa \frac{\mathrm{d}\alpha_r}{\mathrm{d}T_r}\|\nabla z_r\|^2 \left(\frac{\partial T_r}{\partial z_r}\right)^2 \mathrm{d}V, \tag{A 16}$$

$$\int_V \rho_0 g\kappa \|\nabla z_r\|^2 \frac{\partial(\alpha z_r)}{\partial z_r}\frac{\partial T_r}{\partial z_r}\mathrm{d}V = \int_V \rho_0 g\kappa \|\nabla z_r\|^2 \left(1 + z_r \frac{\mathrm{d}\alpha_r}{\mathrm{d}T_r}\frac{\partial T_r}{\partial z_r}\right)\frac{\partial T_r}{\partial z_r}\mathrm{d}V. \tag{A 17}$$

Empirically, it is usually found that $D(APE) > 0$, which is not readily apparent from the form of $D(APE)$, and for which a rigorous mathematical proof remains to be established. Interestingly, while both $W_{r,mixing}$ and B appear to be strongly modified by a temperature-dependent α, this is much less so for their difference $D(APE)$, which is usually found empirically to be well approximated by its 'linear' part $D_L(APE)$. This is important because it clearly establishes that $D(APE)$ and $W_{r,mixing}$ may be significantly different when the temperature dependence of α is retained, in contrast to what is generally admitted based on the L-Boussinesq model. This suggests that results based on the study of the L-Boussinesq model are likely to be more robust and accurate for the description of the KE/APE dynamics than for the description of GPE_r. The condition for $|D_{NL}(APE)| \ll |D_L(APE)|$ to be satisfied is that $\mathrm{d}\alpha_r/\mathrm{d}T_r|\mathrm{d}T_r/\mathrm{d}z_r||z_r - z| \ll 1$, which appears to be satisfied in practice for water or seawater. Whether this is also true for other types of fluids still needs to be established.

Appendix B. Energetics of compressible Navier–Stokes Equations

B.1. *Compressible Navier–Stokes Equations*

This appendix generalizes Winters *et al.* (1995)'s results to the fully compressible Navier–Stokes equations, which are written as

$$\rho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \nabla P = -\rho g\hat{\boldsymbol{z}} + \nabla\cdot\boldsymbol{S}, \tag{B 1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{v} = 0, \tag{B 2}$$

$$\frac{\mathrm{D}\Sigma}{\mathrm{D}t} = \frac{\dot{Q}}{T} = \frac{\rho\varepsilon - \nabla\cdot\boldsymbol{F}_q}{\rho T}, \tag{B 3}$$

$$I = I(\Sigma, \upsilon), \tag{B 4}$$

$$T = T(\Sigma,\upsilon) = \frac{\partial I}{\partial\Sigma}, \qquad P = P(\Sigma,\upsilon) = -\frac{\partial I}{\partial\upsilon}. \tag{B 5}$$

In the present description, the three-dimensional Eulerian velocity field $\boldsymbol{v} = (u, v, w)$, the specific volume $\upsilon = 1/\rho$ (with the density ρ) and the specific entropy Σ are taken as the dependent variables, with the thermodynamic pressure P and the *in situ* temperature T being diagnostic variables in (B 4) and (B 5), where I is the specific internal energy, regarded as a function of Σ and υ. More useful notations include: $\mathrm{D}/\mathrm{D}t = \partial/\partial t + (\boldsymbol{v}\cdot\nabla)$ is the substantial derivative, ε is the dissipation rate of kinetic energy, $\boldsymbol{F}_q = -k_T \rho C_p \nabla T$ is the diffusive heat flux, C_p is the specific heat capacity at constant pressure, k_T is the molecular diffusivity for temperature, g is the acceleration of gravity and $\hat{z}$ is a normal unit vector pointing upward. Moreover, $\boldsymbol{S}$ is the deviatoric stress tensor:

$$\boldsymbol{S}_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left(\lambda - \frac{2\mu}{3} \right) \delta_{ij} \frac{\partial u_\ell}{\partial x_\ell} \tag{B 6}$$

in the classical tensorial notation (i.e. Landau & Lifshitz 1987), where Einstein's summation convention for repeated indices has been adopted and δ_{ij} is the Kronecker delta. The parameters μ and λ are the shear and the bulk (or volume) viscosity respectively.

B.2. *Standard energetics*

The derivation of evolution equations for the standard forms of energy in the context of the fully compressible Navier–Stokes equations is a standard exercise (e.g. de Groot & Mazur 1962; Landau & Lifshitz 1987; Griffies 2004), so only the final results are given. In the standard description of energetics, only the volume-integrated KE, GPE and IE are considered:

$$KE = \int_V \rho \frac{\boldsymbol{v}^2}{2} \, \mathrm{d}V, \qquad GPE = \int_V \rho g z \, \mathrm{d}V \qquad IE = \int_V \rho I(\Sigma, \upsilon) \, \mathrm{d}V, \tag{B 7}$$

whose standard evolution equations are respectively given by

$$\frac{\mathrm{d}KE}{\mathrm{d}t} = -\underbrace{\int_V \rho g w \, \mathrm{d}V}_{W} + \underbrace{\int_V P \frac{\mathrm{D}\upsilon}{\mathrm{D}t}}_{B} \mathrm{d}m + G(KE) - D(KE) - P_a \frac{\mathrm{d}V_{ol}}{\mathrm{d}t}, \tag{B 8}$$

$$\frac{\mathrm{d}GPE}{\mathrm{d}t} = \underbrace{\int_V \rho g w \, \mathrm{d}V}_{W}, \tag{B 9}$$

$$\frac{\mathrm{d}IE}{\mathrm{d}t} = \int_V \rho \dot{Q} \, \mathrm{d}V - \underbrace{\int_V P \frac{\mathrm{D}\upsilon}{\mathrm{D}t} \mathrm{d}m}_{B} = D(KE) + Q_{heating} - Q_{cooling} - B, \tag{B 10}$$

where $G(KE)$ is the rate of work done by the mechanical sources of energy on the fluid, $Q_{heating}$ (respectively $Q_{cooling}$) is the surface-integrated rate of heating (respectively cooling) due to the thermodynamic sources of energy, $\mathrm{d}m = \rho \mathrm{d}V$ and V_{ol} is the total volume of the fluid; additional definitions and justifications are given below. Summing (B 8)–(B 10) yields the following evolution equation for the total energy, $TE = KE + GPE + IE$:

$$\frac{\mathrm{d}TE}{\mathrm{d}t} = G(KE) + Q_{heating} - Q_{cooling} - P_a \frac{\mathrm{d}V_{ol}}{\mathrm{d}t}, \tag{B 11}$$

which states that the total energy of the fluid is modified by the rate of

(*a*) work done by the mechanical sources of energy;

(*b*) heating/cooling done by the thermodynamic sources of energy; and

(*c*) work done by the atmospheric pressure P_a against the volume changes of the fluid.

As these derivations are quite standard, justifications for the above equations are briefly outlined. Thus, the KE equation (B 8) is classically obtained by multiplying the momentum equation by $\boldsymbol{v}$ and integrating over the volume domain. The term W results from the product of $\boldsymbol{v}$ with the gravitational force vector, whereas the product $\boldsymbol{v}\cdot\nabla P=\nabla\cdot(P\boldsymbol{v})-P\nabla\cdot\boldsymbol{v}=\nabla\cdot(P\boldsymbol{v})-(P/\upsilon)\mathrm{D}\upsilon/\mathrm{D}t$ yields the work of expansion/contraction minus the work done by the atmospheric pressure against the total volume changes. The product of the velocity vector with the stress tensor is written as the sum $G(KE)-D(KE)$, where $G(KE)$ represents the work input due to the external stress and $D(KE)$ represents the positive dissipation of kinetic energy. The general expression for the mechanical energy input is

$$G(KE)=\int_{\partial V}\boldsymbol{v}\boldsymbol{S}\cdot\boldsymbol{n}\,\mathrm{d}S=\int_{\partial V}\tau\cdot\boldsymbol{v}\,\mathrm{d}S, \tag{B 12}$$

where $\boldsymbol{v}\boldsymbol{S}$ is the vector with components $(\boldsymbol{S}\boldsymbol{v})_j=S_{ij}u_i$, while $\boldsymbol{S}\boldsymbol{n}=\tau$ is the stress applied along the surface boundary enclosing the fluid. $G(KE)$ is therefore the work of the applied stress done against the fluid velocity. If one assumes a no-slip boundary condition on all solid boundaries, then this work is different from zero only on the free surface. The function $D(KE)$ is the dissipation function

$$D(KE)=\int_V\left\{\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\frac{\partial u_\ell}{\partial x_\ell}\right)^2+\lambda(\nabla\cdot\boldsymbol{v})^2\right\}\mathrm{d}V, \tag{B 13}$$

where, again, the summation convention for repeated indices has been used (e.g. Landau & Lifshitz 1987). The equation for GPE (B 9) is simply obtained by taking the time derivative of its definition, using the fact that $\mathrm{D}(\rho gz\mathrm{d}V)/\mathrm{D}t=\rho gw$, since $\mathrm{D}(\rho\mathrm{d}V)/\mathrm{D}t=0$ from mass conservation. The equation for IE (B 10) results from the fact that the differential of internal energy in the entropy/specific volume representation is given by $\mathrm{d}I=T\mathrm{d}\Sigma-P\mathrm{d}\upsilon$. The terms $Q_{heating}$ and $Q_{cooling}$ represent the surface-integrated net heating and cooling, respectively, going through the surface enclosing the domain.

B.3. *Available energetics*

In this paragraph, we seek to derive separate evolution equations for the available and un-available parts of the total potential energy $PE=IE+GPE+P_aV_{ol}$, as initially proposed by Lorenz (1955), building upon ideas going back to Margules (1903). Specifically, PE is decomposed as follows:

$$\begin{aligned}PE&=\int_V\rho[I(\Sigma,\upsilon)+gz]\,\mathrm{d}V+P_aV_{ol}\\&=\underbrace{\int_V\rho[I(\Sigma,\upsilon)+gz]\,\mathrm{d}V-\int_V\rho[I(\Sigma,\upsilon_r)+gz_r]\,\mathrm{d}V+P_a(V_{ol}-V_{ol,r})}_{APE}\\&\quad+\underbrace{\int_V\rho\,[I(\Sigma,\upsilon_r)+gz_r]\,\mathrm{d}V+P_aV_{ol,r}}_{PE_r},\end{aligned} \tag{B 14}$$

where PE_r is the potential energy of Lorenz (1955)'s reference state and $APE=PE-PE_r$ is the available potential energy. As is well known, the reference state is the state minimizing the total potential energy of the system in an adiabatic rearrangement of the fluid parcels. From a mathematical viewpoint, Lorenz (1955)'s reference state can be defined in terms of a mapping taking a parcel located at $(\boldsymbol{x},t)$ in the given state to its position $(\boldsymbol{x}_r,t)$ in the reference state, such that the mapping preserves the specific entropy Σ and mass $\rho\mathrm{d}V$ of the parcel, viz.,

$$\Sigma(\boldsymbol{x},t)=\Sigma(\boldsymbol{x}_r,t)=\Sigma_r(z_r,t), \tag{B 15}$$

$$\rho(\boldsymbol{x},t)\mathrm{d}V=\rho(\boldsymbol{x}_r,t)\mathrm{d}V_r=\rho_r(z_r,t)\mathrm{d}V_r, \tag{B 16}$$

where the second condition can be equivalently formulated in terms of the Jacobian $J = \partial(\boldsymbol{x}_r)/\partial(\boldsymbol{x})$ of the mapping between the actual and reference state as follows:

$$\rho(\boldsymbol{x}, t) = \rho(\boldsymbol{x}_r, t)\frac{\partial(\boldsymbol{x}_r)}{\partial(\boldsymbol{x})} = \rho_r(z_r, t)\frac{\partial(\boldsymbol{x}_r)}{\partial(\boldsymbol{x})}. \tag{B 17}$$

Prior to deriving evolution equation for PE_r and APE, it is useful to mention three important properties of the reference state, namely:

(*a*) The density $\rho_r = \rho_r(z_r, t)$ and the pressure $P_r = P_r(z_r, t)$ of the Lorenz (1955)'s background reference state are functions of z_r alone (and time);

(*b*) The background density ρ_r and the pressure P_r are in hydrostatic balance at all times, i.e. $\partial P_r/\partial z_r = -\rho_r g$ (this is a consequence of the reference state being the state minimizing the total potential energy in an adiabatic rearrangement of the parcels);

(*c*) The velocity $\boldsymbol{v}_r = (\mathrm{D}x_r/\mathrm{D}t, \mathrm{D}y_r/\mathrm{D}t, \mathrm{D}z_r/\mathrm{D}t)$ of the parcels in the reference state satisfies the usual mass conservation equation:

$$\frac{\mathrm{D}\upsilon_r}{\mathrm{D}t} = \upsilon_r \nabla_r \cdot \boldsymbol{v}_r, \tag{B 18}$$

where $\nabla_r \cdot \boldsymbol{v}_r$ is the velocity divergence expressed coordinates system of the reference state, which is a consequence of the mass of the fluid parcels being conserved by the mapping between the actual and the reference states.

Equation (B 18) is important for it allows an easy demonstration of the following result:

$$\begin{aligned}\int_V \rho P_r \frac{\mathrm{D}\upsilon_r}{\mathrm{D}t}\,\mathrm{d}V &= \int_{V_r} \rho_r P_r \frac{\mathrm{D}\upsilon_r}{\mathrm{D}t}\,\mathrm{d}V_r = \int_{V_r} P_r \nabla_r \cdot \boldsymbol{v}_r \,\mathrm{d}V_r \\ &= \int_{\partial V_r} P_r \boldsymbol{v}_r \cdot \boldsymbol{n}_r \,\mathrm{d}S_r - \int_{V_r} \boldsymbol{v}_r \cdot \nabla P_r \,\mathrm{d}V_r = P_a \frac{\mathrm{d}V_{ol,r}}{\mathrm{d}t} + \underbrace{\int_{V_r} \rho_r g w_r \,\mathrm{d}V_r}_{W_r},\end{aligned} \tag{B 19}$$

which establishes the equivalence between the work of expansion and the work against gravity in the reference state, where $\boldsymbol{n}_r$ is an outward-pointing unit vector normal to the boundary ∂V_r enclosing the fluid in the reference state. In (B 19), the first equality stems from expressing the first integral in the reference state; the second equality uses (B 18); the third equality results from an integration by parts; and the final equality stems from that P_r depends on z_r and t only, is in hydrostatic balance and by using the boundary condition $\boldsymbol{v}_r \cdot \boldsymbol{n}_r = w_r = \partial\eta_r/\partial t$ at the surface assumed to be located at $z_r = \eta_r(t)$.

B.4. *Evolution of the background potential energy PE_r*

We seek an evolution equation for the background PE_r by taking the time derivative of the expression in (B 14), which yields

$$\begin{aligned}\frac{\mathrm{d}PE_r}{\mathrm{d}t} &= \int_V \rho\left[T_r\frac{\mathrm{D}\Sigma}{\mathrm{D}t} - P_r\frac{\mathrm{D}\upsilon_r}{\mathrm{D}t} + g w_r\right]\mathrm{d}V + P_a\frac{\mathrm{d}V_{ol,r}}{\mathrm{d}t} + \underbrace{\int_V [I(\Sigma, \upsilon_r) + g z_r]\frac{\mathrm{D}(\rho \mathrm{d}V)}{\mathrm{D}t}}_{=0} \\ &= \int_V \rho T_r \frac{\dot{Q}}{T}\,\mathrm{d}V = \int_V \rho \dot{Q}\,\mathrm{d}V + \int_V \rho\left(\frac{T_r - T}{T}\right)\dot{Q}\,\mathrm{d}V \\ &= \dot{Q}_{net} + (1-\gamma_\varepsilon)D(KE) + D(APE) - G(APE),\end{aligned} \tag{B 20}$$

where the final result was arrived at by making use of (B 19), as well as of the definitions

$$\int_V \rho \dot{Q}\,\mathrm{d}V = \int_V \left\{\nabla\cdot\left(\kappa\rho C_p\nabla T\right) + \rho\varepsilon\right\}\,\mathrm{d}V = \dot{Q}_{net} + D(KE), \tag{B 21}$$

$$\dot{Q}_{net} = \int_S \kappa\rho C_p \nabla T\cdot\boldsymbol{n}\,\mathrm{d}S = Q_{heating} - Q_{cooling}, \tag{B 22}$$

$$G(APE) = \int_S \left(\frac{T-T_r}{T}\right)\kappa\rho C_p\nabla T\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{B 23}$$

$$D(APE) = \int_V \kappa\rho C_p\nabla T\cdot\nabla\left(\frac{T-T_r}{T}\right)\mathrm{d}V, \tag{B 24}$$

$$\gamma_\varepsilon D(KE) = \int_V \left(\frac{T-T_r}{T}\right)\rho\varepsilon\,\mathrm{d}V, \tag{B 25}$$

where $\boldsymbol{n}$ is the unit normal vector pointing outward the domain. Equation (B 22) expresses the net diabatic heating $\dot{Q}_{net}$ due to the surface heat fluxes as the sum of a purely positive $Q_{heating}$ and negative $-Q_{cooling}$ contributions. Equation (B 23) defines the rate of available potential energy produced by the surface heat fluxes. The term $D(APE)$, as defined by (B 24), is physically expected to represent the rate at which APE is dissipated by molecular diffusion, so that it is expected to be positive in general, which has been so far only established empirically using randomly generated temperature fields, but a rigorous mathematical proof is lacking. Finally, (B 25) states that a tiny fraction of the diabatic heating due to viscous dissipation might be recycled to produce work. If γ_ε could be proven to be positive, it could probably be included as part of the $G(APE)$. In the following, it will just be neglected for simplicity.

B.5. *Evolution of APE*

In the previous section, we defined the total potential energy as the sum of GPE, IE and the quantity $P_a V_{ol}$; see (B 14). As a result, using the evolution equations for GPE and IE previously derived, the evolution equation for PE is given by

$$\frac{\mathrm{d}PE}{\mathrm{d}t} = W - B + \dot{Q}_{net} + P_a\frac{\mathrm{d}V_{ol}}{\mathrm{d}t}. \tag{B 26}$$

Now, combining this equation with the one previously derived for PE_r allows us to derive the following equation for the available potential energy, $APE = PE - PE_r$, defined as the difference between the potential energy and its background value:

$$\begin{aligned}\frac{\mathrm{d}APE}{\mathrm{d}t} &\approx \underbrace{W - B + P_a\frac{\mathrm{d}V_{ol}}{\mathrm{d}t}}_{C(KE,APE)} + \dot{Q}_{net} + D(KE) - [D(KE) + D(APE) + \dot{Q}_{net} - G(APE)]\\ &= C(KE,APE) + G(APE) - D(APE),\end{aligned} \tag{B 27}$$

where the final expression neglects the small term $\gamma_\varepsilon D(KE)$. The corresponding energy flowchart for the $KE/APE/PE_r$ system is very simple and is illustrated in figure 10(a). This diagram shows that mechanical energy enters the fluid via the KE reservoir and thermal energy enters it via the PE_r reservoir. There are two dissipation routes associated with the viscous dissipation of KE and the diffusive dissipation of APE. Only a certain part $G(APE)$ of the thermodynamic energy input can be converted into APE and hence into KE, which is processed via the PE_r reservoir. The two-headed arrow indicates the reversible conversion between KE and APE.

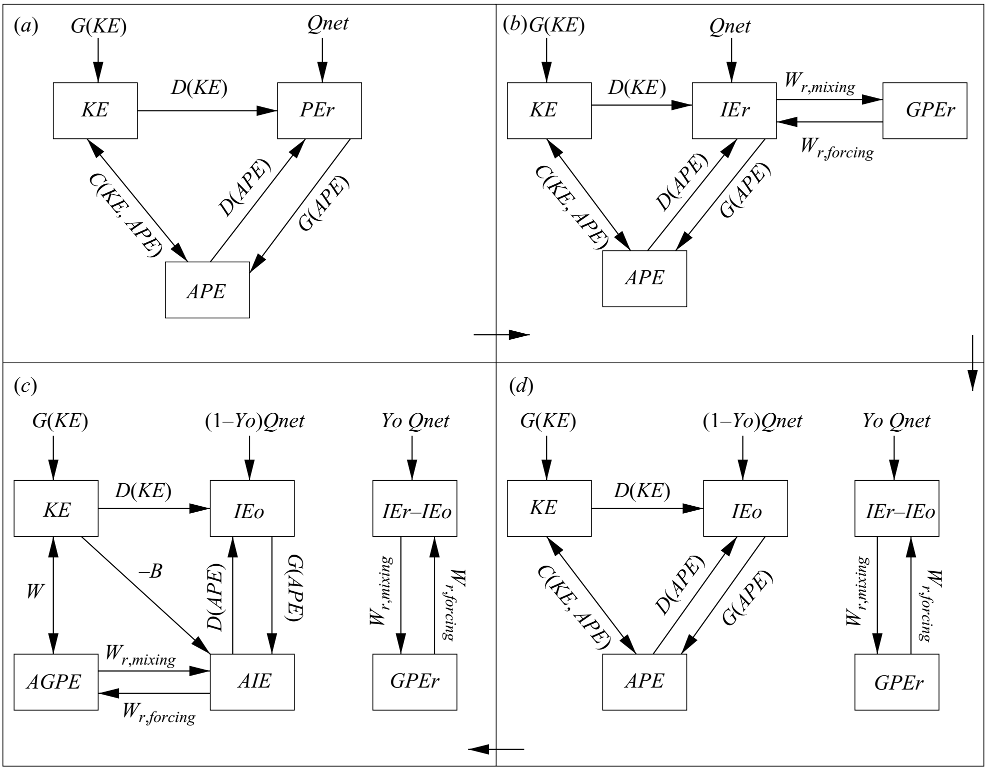

FIGURE 10. Successive refinements of the energetics of a forced/dissipated stratified fluid. (*a*) The $KE/APE/PE_r$ representation. (*b*) Decomposition of PE_r into $IE_r + GPE_r$. (*c*) Decomposition of IE_r into a dead part IE_0 and an exergy part $IE_{exergy} = IE_r - IE_0$. (*d*) Decomposition of APE into AIE and $AGPE$, revealing the link between $C(KE, APE)$ to the density flux W and the work of expansion/contraction B.

B.6. *Splitting of PE_r into GPE_r and IE_r*

Although the $KE/APE/PE_r$ system offers a simple picture of the energetics of a (turbulent or not) stratified fluid, it is useful to further decompose the background PE_r reservoir into its GPE and IE components, in order to establish the link with the existing literature about turbulent mixing, as well as with Munk & Wunsch (1998)'s theory. The particular question to be addressed is to understand how much of $D(KE)$ and $D(APE)$ are actually spread over GPE_r and IE_r. Likewise, to what extent do $\dot{Q}_{net}$ and $G(APE)$ affect GPE_r compared with IE_r, where we have the following definitions:

$$IE_r = \int_V \rho(\boldsymbol{x}, t) I(\Sigma, \upsilon_r)\,\mathrm{d}V = \int_{V_r} \rho_r(z_r, t) I(\Sigma, \upsilon_r)\,\mathrm{d}V_r, \qquad (\mathrm{B}\,28)$$

$$GPE_r = \int_V \rho(\boldsymbol{x}, t) g z_r(\boldsymbol{x}, t)\,\mathrm{d}V = \int_{V_r} \rho_r(z_r, t) g z_r\,\mathrm{d}V_r, \qquad (\mathrm{B}\,29)$$

by expressing the integrals in the coordinate system associated with either the actual state or the reference state. By definition

$$\frac{\mathrm{d}\,GPE_r}{\mathrm{d}t} = \int_V \rho g \frac{Dz_r}{Dt}\,\mathrm{d}V + \underbrace{\int_V g z_r \frac{D(\rho \mathrm{d}V)}{\mathrm{d}t}}_{=0} = \int_V \rho g w_r\,\mathrm{d}V = W_r, \qquad (\mathrm{B}\,30)$$

so that the evolution equation for $IE_r + P_a V_{ol,r} = PE_r - GPE_r$ simply is

$$\frac{\mathrm{d}(IE_r + P_a V_{ol,r})}{\mathrm{d}t} = \dot{Q}_{net} + D(KE) + D(APE) - G(APE) - W_r. \tag{B 31}$$

In order to make progress, we need to relate W_r to the different sources and sinks affecting PE_r, as identified in figure (10). To that end, we use the fact that W_r is related to the work of expansion in the reference state, as shown by (B 19), and regard $\upsilon = \upsilon(\Sigma, P)$ as a function of entropy and pressure, for which the total differential is given by

$$\mathrm{d}\upsilon = \Gamma \mathrm{d}\Sigma - \frac{1}{\rho^2 c_s^2}\mathrm{d}P, \tag{B 32}$$

where $\Gamma = \alpha T/(\rho C_p)$ is the so-called adiabatic lapse rate (e.g. Feistel 2003) and $c_s^2 = (\partial P/\partial \rho)_\Sigma$ is the squared sound of speed. As a result, the expression for W_r becomes

$$W_r = \int_{V_r} P_r \frac{\mathrm{D}\upsilon_r}{\mathrm{D}t}\rho_r \,\mathrm{d}V_r - P_a \frac{\mathrm{d}V_{ol,r}}{\mathrm{d}t} = \int_{V_r} P_r'\rho_r \left[\frac{\alpha_r T_r}{\rho_r C_{pr}}\frac{\dot{Q}}{T} - \frac{1}{\rho_r^2 c_{sr}^2}\frac{\mathrm{D}P_r}{\mathrm{D}t} \right] \mathrm{d}V_r, \tag{B 33}$$

where $P_r' = P_r - P_a$ is the pressure corrected by the atmospheric pressure, by noting that we have

$$\frac{\mathrm{d}V_{ol,r}}{\mathrm{d}t} = \int_V \frac{\mathrm{D}\upsilon_r}{\mathrm{D}t}\rho \,\mathrm{d}V = \int_{V_r} \frac{\mathrm{D}\upsilon_r}{\mathrm{D}t}\rho_r \,\mathrm{d}V_r. \tag{B 34}$$

In order to simplify (B 33), let us recall that mass conservation can be rewritten in hydrostatic pressure coordinates as follows:

$$\nabla_r \cdot \boldsymbol{u}_r + \frac{\partial}{\partial P_r}\frac{\mathrm{D}P_r}{\mathrm{D}t} = 0 \tag{B 35}$$

(e.g. Haltiner & Williams 1980; de Szoeke & Samelson 2002). As a result, it follows that integrating (B 35) from the surface where $P_a = cst$, and hence where $\mathrm{D}P_r/\mathrm{D}t = 0$, to an arbitrary level indicates that the surface integral of $\mathrm{D}P_r/\mathrm{D}t$ must vanish along any $z_r = cst$ surfaces. As a consequence, the term depending on $\mathrm{D}P_r/\mathrm{D}t$ in (B 33) must vanish. For an alternative derivation of this result, see Pauluis (2007). The remaining term can be written as follows:

$$\begin{aligned}
W_r &= \int_V \frac{P_r'\alpha_r T_r}{\rho_r C_{pr} T}\{\nabla\cdot(\kappa\rho C_p \nabla T) + \rho\varepsilon\}\,\mathrm{d}V \\
&= \int_V \frac{P_r'\alpha_r}{\rho_r C_{pr}}\left(1 + \frac{T_r - T}{T}\right)\nabla\cdot(\kappa\rho C_p\nabla T)\,\mathrm{d}V + \int_V \frac{P_r'\alpha_r T_r}{\rho_r C_{pr} T}\rho\varepsilon\,\mathrm{d}V \\
&= \underbrace{\int_V \frac{P_r'\alpha_r}{\rho_r C_{pr}}\nabla\cdot(\kappa\rho C_p\nabla T)\,\mathrm{d}V}_{W_{r,mixing} - W_{r,forcing}} \\
&\quad + \underbrace{\int_V \frac{P_r'\alpha_r}{\rho_r C_{pr}}\left(\frac{T_r - T}{T}\right)\nabla\cdot(\kappa\rho C_p\nabla T)\,\mathrm{d}V}_{\Upsilon_{r,ape}D(APE)} + \underbrace{\int_V \frac{P_r'\alpha_r T_r}{\rho_r C_{pr} T}\rho\varepsilon\,\mathrm{d}V}_{\Upsilon_{r,ke}D(KE)} \\
&= W_{r,mixing} - W_{r,forcing} + \Upsilon_{r,ape}D(APE) + \Upsilon_{r,ke}D(KE),
\end{aligned} \tag{B 36}$$

where we defined

$$W_{r,mixing} = -\int_V \kappa\rho C_p \nabla T \cdot \nabla\left(\frac{\alpha_r P_r'}{\rho_r C_{pr}}\right) \mathrm{d}V, \tag{B 37}$$

$$W_{r,forcing} = -\int_S \frac{\alpha_r P_r'}{\rho_r C_{pr}} \kappa\rho C_p \nabla T \cdot \boldsymbol{n}\, \mathrm{d}S, \tag{B 38}$$

$$\varUpsilon_{r,ape} D(APE) = \int_V \frac{\alpha_r P_r'}{\rho_r C_{pr}} \left(\frac{T_r - T}{T}\right) \nabla \cdot (\kappa\rho C_p \nabla T)\, \mathrm{d}V, \tag{B 39}$$

$$\varUpsilon_{r,ke} D(KE) = \int_V \frac{\alpha_r P_r}{\rho_r C_{pr}} \frac{T_r}{T} \rho\varepsilon\, \mathrm{d}V. \tag{B 40}$$

Equation (B 36) shows that the variations of GPE_r are affected by the following:

(*a*) Turbulent mixing associated with $W_{r,mixing}$. This expression is similar to the one previously derived for the L-Boussinesq model. The classical Boussinesq expression can be recovered from using the approximation $T \approx T_r$, taking α_r, ρ_r and C_{pr} as constants, and using the approximation $P_r \approx = -\rho_0 g z_r$, which yields

$$W_{r,mixing} \approx \int_V \kappa\rho_0 g \|\nabla z_r\|^2 \alpha \frac{\partial T_r}{\partial z_r}\, \mathrm{d}V.$$

(*b*) The surface forcing associated with $W_{r,forcing}$. Likewise, the L-Boussinesq expression can be recovered by making the same approximation, yielding

$$W_{r,forcing} \approx \int_V \frac{\alpha g z_r}{C_p} Q_{surf}\, \mathrm{d}S.$$

Note that in the L-Boussinesq approximation, we have

$$W_{r,forcing} \approx G(APE),$$

which is not generally true in the fully compressible Navier–Stokes equation.

(*c*) The contribution from the viscous and the diffusive dissipation of KE and APE respectively associated with $D(KE)$ and $D(APE)$. Note that the coefficients $\varUpsilon_{r,ape}$ and $\varUpsilon_{r,ke}$ are very small for a nearly incompressible fluid such as seawater. For instance, typical values are $\alpha = 10^{-4}\,\mathrm{K}^{-1}$, $P = 4 \times 10^7\,\mathrm{Pa}$, $\rho = 10^3\,\mathrm{kg\ m^{-3}}$ and $C_p = 4 \times 10^3\,\mathrm{J\ kg^{-1}K^{-1}}$, which yield

$$\varUpsilon_r = O\left(\frac{10^{-4} \times 4 \times 10^7}{10^3 \times 4 \times 10^3}\right) = O(10^{-3}).$$

From this, it follows that at leading order, the direct effects of $D(APE)$ and $D(KE)$ on GPE_r can be safely neglected compared to the other two effects, so that

$$\frac{\mathrm{d}\, GPE_r}{\mathrm{d}t} = W_r \approx W_{r,mixing} - W_{r,forcing}. \tag{B 41}$$

The resulting modifications to the energy flowchart are displayed in figure 10(*b*). At leading order, the effects of the forcing and the mixing on GPE_r appear as conversion terms with IE_r.

B.7. *Further partitioning of internal energy into a ‘dead’ and ‘exergy’ component*

As seen previously, the L-Boussinesq model is such that

$$D(APE) \approx W_{r,mixing}, \qquad G(APE) \approx W_{r,forcing}, \tag{B 42}$$

which may give the impression, based on figure 10(*b*), that the APE dissipated by $D(APE)$ is actually converted into GPE_r, while $G(APE)$ may also appear as originating from GPE_r. The purpose of the following is to show that this is actually not the case. To that end, we introduce an equivalent isothermal state having exactly the same energy as that of Lorenz (1955)'s reference state, that is defined by

$$\underbrace{IE_r + GPE_r + P_a V_{ol,r}}_{PE_r} = \underbrace{IE_0 + GPE_0 + P_a V_{ol,0}}_{PE_0}. \tag{B 43}$$

Because both Lorenz's reference state and the equivalent thermodynamic equilibrium state are in hydrostatic balance at all times, PE_r and PE_0 are just the total enthalpies of the two states. This makes it possible to define each parcel by its horizontal coordinates (x, y) and hydrostatic pressure P and assume that the dead state can be obtained from Lorenz's reference state by an isobaric process, so that $(x_0, y_0, P_0) = (x_r, y_r, P_r)$, which in turn implies $(\mathrm{d}x_0/\mathrm{d}t, \mathrm{d}y_0/\mathrm{d}t, \mathrm{d}P_0/\mathrm{d}t) = (\mathrm{d}x_r/\mathrm{d}t, \mathrm{d}y_r/\mathrm{d}t, \mathrm{d}P_r/\mathrm{d}t)$.

Before looking at the evolution of the dead state, let us establish that if the pressure P is in hydrostatic balance at all times, then we have the following result:

$$\int_V \frac{\mathrm{D}P}{\mathrm{D}t}\,\mathrm{d}V = \int_V \boldsymbol{u}\cdot\nabla_h P\,\mathrm{d}V, \tag{B 44}$$

where $\boldsymbol{u}$ is the horizontal part of the three-dimensional velocity field and ∇_h is the horizontal nabla operator. The proof reads

$$\begin{aligned}
\int_V \left(\frac{\mathrm{D}P}{\mathrm{D}t} - \boldsymbol{u}\cdot\nabla_h P\right)\mathrm{d}V &= \int_V \left(\frac{\partial P}{\partial t} + w\frac{\partial P}{\partial z}\right)\mathrm{d}V \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_V P\,\mathrm{d}V - P_a\frac{\mathrm{d}V_{ol}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho g z\,\mathrm{d}V \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left\{P_a V_{ol} + M_{tot} g H_b + \int_V \rho g z\,\mathrm{d}V\right\} - P_a\frac{\mathrm{d}V_{ol}}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho g z\,\mathrm{d}V = 0,
\end{aligned}$$

where V_{ol} and M_{tot} are the total volume and mass of the fluid, whose expressions are

$$V_{ol} = \int_S (\eta(x, y, t) + H_b)\,\mathrm{d}x\,\mathrm{d}y, \qquad gM_{tot} = \int_S (P_b(x, y, t) - P_a)\,\mathrm{d}x\,\mathrm{d}y,$$

where $z = \eta(x, y, t)$ is the equation for the free surface, $P_b(x, y, t)$ is the bottom pressure, H_b is the total depth of the basin and the expression in parentheses was obtained by using the following result:

$$\begin{aligned}
\int_V P\mathrm{d}V &= \int_S [Pz]_{-H_b}^{\eta}\,\mathrm{d}x\,\mathrm{d}y + \int_V \rho g z\,\mathrm{d}V = \int_S [P_a\eta + P_b H_b]\,\mathrm{d}x\,\mathrm{d}y + \int_V \rho g z\,\mathrm{d}V \\
&= \underbrace{P_a\int_S (\eta + H_b)\,\mathrm{d}x\,\mathrm{d}y}_{P_a V_{ol}} + \underbrace{H_b\int_S (P_b - P_a)\,\mathrm{d}x\,\mathrm{d}y}_{M_{tot} g H_b} + \int \rho g z\,\mathrm{d}V.
\end{aligned}$$

The important consequence of (B 44) is that the volume integral of $\mathrm{D}P/\mathrm{D}t$ identically vanishes when P is independent of the horizontal coordinates, as is the case for P_r and P_0. Now, using the expression for the enthalpy $I + P/\rho$:

$$\mathrm{d}(I + P/\rho) = C_p\mathrm{d}T + \left(\upsilon - T\frac{\partial \upsilon}{\partial T}\right)\mathrm{d}P = C_p\mathrm{d}T + \upsilon(1 - \alpha T)\,\mathrm{d}P,$$

we can derive the following equation for PE_0:

$$\frac{\mathrm{d}\,PE_0}{\mathrm{d}t} = \int_{V_0} \rho_0 \left(C_{p0} \frac{\mathrm{D}T_0}{\mathrm{D}t} + (1-\alpha_0 T_0)\frac{\mathrm{D}P_0}{\mathrm{D}t} \right) \mathrm{d}V_0 = \frac{\mathrm{d}T_0}{\mathrm{d}t} \int_{V_0} \rho_0 C_{p0}\, \mathrm{d}V_0, \qquad \text{(B 45)}$$

which naturally provides the following equation for T_0:

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = \frac{D(KE) + D(APE) + \dot{Q} - G(APE)}{\int_{V_0} \rho_0 C_{p0}\, \mathrm{d}V_0}. \qquad \text{(B 46)}$$

We can now derive an evolution equation for GPE_0, using the relation

$$\frac{\mathrm{d}\,GPE_0}{\mathrm{d}t} = \int_{V_0} P_0' \frac{\mathrm{D}\upsilon_0}{\mathrm{D}t} \rho_0\, \mathrm{d}V_0, \qquad \text{(B 47)}$$

where $P_0' = P_0 - P_a$. Now, expressing $\mathrm{d}\upsilon = \upsilon\alpha \mathrm{d}T + \upsilon\gamma \mathrm{d}P$, where γ is the isothermal expansion coefficient, we arrive at the following expression:

$$\frac{\mathrm{d}\,GPE_0}{\mathrm{d}t} = \int_{V_0} \rho_0 P_0' \left[\upsilon_0 \frac{\mathrm{D}T_0}{\alpha_0 \mathrm{D}t} - \upsilon_0 \gamma_0 \frac{\mathrm{D}P_0}{\mathrm{D}t} \right] \mathrm{d}V_0 = \frac{\mathrm{d}T_0}{\mathrm{d}t} \int_{V} \alpha_0 P_0' d V_0, \qquad \text{(B 48)}$$

noting again that the term proportional to $\mathrm{D}P_0/\mathrm{D}t$ must vanish from the arguments developed above, so that we simply have

$$\frac{\mathrm{d}\,GPE_0}{\mathrm{d}t} = \Upsilon_0 [D(KE) + D(APE) + \dot{Q}_{net} - G(APE)], \qquad \text{(B 49)}$$

where

$$\Upsilon_0 = \frac{\displaystyle\int_{V_0} P_0' \alpha_0\, \mathrm{d}V_0}{\displaystyle\int_{V_0} \rho_0 C_{p0}\, \mathrm{d}V_0}. \qquad \text{(B 50)}$$

As a result, it follows that

$$\begin{aligned} \frac{\mathrm{d}(IE_0 + P_a V_{ol,0})}{\mathrm{d}t} &= \frac{\mathrm{d}(PE_0 - GPE_0)}{\mathrm{d}t} \\ &= (1-\Upsilon_0)[D(KE) + D(APE) + \dot{Q}_{net} - G(APE)]. \qquad \text{(B 51)} \end{aligned}$$

Let us now define the exergy part of the $IE_r + P_a V_{ol,r}$ as

$$IE_{exergy} = IE_r - IE_0 + P_a (V_{ol,r} - V_{ol,0}). \qquad \text{(B 52)}$$

The equation is

$$\begin{aligned} \frac{\mathrm{d}\,IE_{exergy}}{\mathrm{d}t} &= -W_r + \Upsilon_0 [D(KE) + D(APE) + \dot{Q}_{net} - G(APE)] \\ &= (\Upsilon_0 - \Upsilon_{r,ke}) D(KE) + (\Upsilon_0 - \Upsilon_{r,ape}) D(APE) + \Upsilon_0 [\dot{Q}_{net} - G(APE)] \\ &\quad - W_{r,mixing} + W_{r,forcing}. \qquad \text{(B 53)} \end{aligned}$$

Again, neglecting the terms proportional to $\alpha P/(\rho C_p)$ yields the following simplification:

$$\frac{\mathrm{d}(IE_0 + P_a V_{ol,0})}{\mathrm{d}t} \approx D(APE) + D(KE) + (1-\Upsilon_0)\dot{Q}_{net} - G(APE), \qquad \text{(B 54)}$$

$$\frac{\mathrm{d}(IE_{exergy})}{\mathrm{d}t} \approx W_{r,mixing} - W_{r,forcing} + \Upsilon_0 \dot{Q}_{net}. \qquad \text{(B 55)}$$

The corresponding energy flowchart is illustrated in figure 10(d). This figure shows that when IE_r is decomposed into its dead and exergy parts, a decoupling between the $KE/APE/IE_0$ and the $IE_r - IE_0/GPE_r$ reservoirs appears at leading order. Note, however, that the rates between the reservoirs remain coupled, owing to the correlation between $D(APE)$ and $W_{r,mixing}$, as well as between $G(APE)$ and $W_{r,forcing}$ discussed in this paper, and which is the central topic of turbulent mixing theory.

B.8. *Separate evolution of APE into GPE and IE components*

We conclude the evolution equation of energetics by further splitting the APE reservoir into its GPE and IE components. Using the previous relations shows that

$$\frac{\mathrm{d}\,AGPE}{\mathrm{d}t} = W - W_r \approx W - W_{r,mixing} + W_{r,forcing}, \tag{B 56}$$

$$\begin{aligned}\frac{\mathrm{d}\,AIE}{\mathrm{d}t} &\approx W - B + G(APE) - D(APE) - [W - W_{r,mixing} + W_{r,forcing}] \\ &\approx -B + W_{r,mixing} - D(APE) + G(APE) - W_{r,forcing}.\end{aligned} \tag{B 57}$$

For seawater, it is generally found that AIE accounts for around 10 % of the total APE, and hence to a good approximation $APE \approx AGPE$, which is implicit in the Boussinesq approximation. Equating $d(AGPE)/\mathrm{d}t$ with $d(APE)/\mathrm{d}t$ amounts to requiring that $d(AIE)/\mathrm{d}t \approx 0$. By imposing that the forcing and the mixing terms vanish separately, one obtains

$$D(APE) \approx W_{r,mixing} - B, \tag{B 58}$$

$$G(APE) \approx W_{r,forcing}, \tag{B 59}$$

which are equivalent to those of the L-Boussinesq and the NL-Boussinesq models. The corresponding energy flowchart is depicted in figure 10(c). The key feature of this figure is to reveal that the conversion rates between $AGPE$ and AIE are identical to those taking place between $IE_r - IE_0$ and GPE_r, where the coupling appears to occur fundamentally between stirring and mixing.

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